

ZEROS AND FACTORIZATIONS OF QUATERNION POLYNOMIALS: THE ALGORITHMIC APPROACH

VLADIMIR BOLOTNIKOV

ABSTRACT. It is known that polynomials over quaternions may have spherical zeros and isolated left and right zeros. These zeros along with appropriately defined multiplicities form the zero structure of a polynomial. In this paper, we equivalently describe the zero structure of a polynomial in terms of its left and right spherical divisors as well as in terms of left and right indecomposable divisors. Several algorithms are proposed to find left/right zeros and left/right spherical divisors of a quaternion polynomial, to construct a polynomial with prescribed zero structure and more generally, to construct the least left/right common multiple of given polynomials. Similar questions are briefly discussed in the setting of quaternion formal power series.

1. INTRODUCTION

Any monic complex polynomial p of positive degree can be represented as

$$p = \prod_{k=1}^m \rho_{\alpha_k}^{n_k} = \text{lcm}(\rho_{\alpha_1}^{n_1}, \dots, \rho_{\alpha_m}^{n_m}), \quad \rho_{\alpha_k} := z - \alpha_k; \quad \alpha_k \in \mathbb{C}, \quad n_k \in \mathbb{N}, \quad (1.1)$$

where $\alpha_1, \dots, \alpha_m$ are *distinct zeros* of p of respective multiplicities n_1, \dots, n_m and where **lcm** means the least common multiple. The set $\{(\alpha_k, n_k)\}_{k=1}^m$ may be referred to as to the *zero structure* of p and it is clear that a monic polynomial is uniquely recovered from its zero structure by formulas (1.1). Since the second representation in (1.1) is equivalent to the primary ideal decomposition $\langle p \rangle = \bigcap_{k=1}^m \langle \rho_{\alpha_k}^{n_k} \rangle$ for the ideal $\langle p \rangle$ generated by p and since this decomposition is unique, the zero structure of p can be alternatively defined as the collection of generators of irreducible components in the primary ideal decomposition of $\langle p \rangle$.

The case of polynomials over a division ring (rather than a field) is not that simple [21], [17], [11]. Here we focus on polynomials over the skew field \mathbb{H} of quaternions

$$\alpha = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3 \quad (x_0, x_1, x_2, x_3 \in \mathbb{R}), \quad (1.2)$$

where the imaginary units $\mathbf{i}, \mathbf{j}, \mathbf{k}$ commute with \mathbb{R} and satisfy $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. For $\alpha \in \mathbb{H}$ of the form (1.2), its real and imaginary parts, the quaternion conjugate and the absolute value are defined as $\text{Re}(\alpha) = x_0$, $\text{Im}(\alpha) = \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3$, $\bar{\alpha} = \text{Re}(\alpha) - \text{Im}(\alpha)$ and $|\alpha|^2 = \alpha\bar{\alpha} = |\text{Re}(\alpha)|^2 + |\text{Im}(\alpha)|^2$, respectively.

Two quaternions α and β are called *equivalent* (conjugate to each other) if $\alpha = h^{-1}\beta h$ for some nonzero $h \in \mathbb{H}$; in notation, $\alpha \sim \beta$. It turns out (see e.g., [5]) that

$$\alpha \sim \beta \quad \text{if and only if} \quad \text{Re}(\alpha) = \text{Re}(\beta) \quad \text{and} \quad |\alpha| = |\beta|. \quad (1.3)$$

Hence, the *conjugacy class* of a given $\alpha \in \mathbb{H}$ form a 2-sphere (of radius $|\operatorname{Im}(\alpha)|$ around $\operatorname{Re}(\alpha)$) which will be denoted by $[\alpha]$. A finite ordered collection $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_k)$ will be called a *spherical chain* (of the length k) if

$$\alpha_1 \sim \alpha_2 \sim \dots \sim \alpha_k \quad \text{and} \quad \alpha_{j+1} \neq \bar{\alpha}_j \quad \text{for} \quad j = 1, \dots, k-1. \quad (1.4)$$

We let $\mathbb{H}[z]$ be the ring of polynomials in one formal variable z which commutes with quaternionic coefficients. Multiplication in $\mathbb{H}[z]$ (as well as in \mathbb{H}) is not commutative; however, any (left or right) ideal in $\mathbb{H}[z]$ is principal. We will use notation

$$\langle p \rangle_{\mathbf{r}} := \{pq : q \in \mathbb{H}[z]\} \quad \text{and} \quad \langle p \rangle_{\mathbf{l}} := \{qp : q \in \mathbb{H}[z]\}$$

for respectively the right and the left ideal generated by p . The subscript will be dropped if the ideal is two-sided; it is not hard to show that any two-sided ideal in $\mathbb{H}[z]$ is generated by a polynomial with real coefficients.

For any non-constant polynomial $f \in \mathbb{H}[z]$, the sets $\mathcal{Z}_{\mathbf{l}}(f)$ and $\mathcal{Z}_{\mathbf{r}}(f)$ of its left and right zeros (see Section 2.1 for definitions) are non-empty and are contained in a finite union of distinct conjugacy classes:

$$\mathcal{Z}(f) := \mathcal{Z}_{\mathbf{l}}(f) \cup \mathcal{Z}_{\mathbf{r}}(f) \subset \bigcup V_i.$$

Moreover, each conjugacy class V_i either contains exactly one left and one right zero of f or $V_i \subset \mathcal{Z}_{\mathbf{l}}(f) \cap \mathcal{Z}_{\mathbf{r}}(f)$; in the latter case, we say that V_i is the *spherical zero* of f . Observe that the set of all polynomials $f \in \mathbb{H}[z]$ having the spherical zero V is the two-sided ideal generated by the real polynomial

$$\mathcal{K}_V(z) = (z - \alpha)(z - \bar{\alpha}) = z^2 - 2z \cdot \operatorname{Re}(\alpha) + |\alpha|^2, \quad \alpha \in V \quad (1.5)$$

(the *characteristic polynomial* of V); it follows from characterization (1.3) that α in (1.5) can be replaced by any other element in V .

Several algorithms for finding left/right roots (by means of complex root-finding for a polynomial with real coefficients are known [16], [20], [13], [14]. Yet another algorithm of this type (Algorithm 3.4 below) is our contribution to the topic.

By the division algorithm, any non-constant monic $f \in \mathbb{H}[z]$ can be factored as

$$f(z) = (z - \gamma_1)(z - \gamma_2) \cdots (z - \gamma_N), \quad \gamma_1, \dots, \gamma_N \in \mathbb{H}, \quad (1.6)$$

where $\gamma_1 \in \mathcal{Z}_{\mathbf{l}}(f)$ and $\gamma_N \in \mathcal{Z}_{\mathbf{r}}(f)$. Although it may happen that none of $\gamma_2, \dots, \gamma_{N-1}$ belongs to $\mathcal{Z}(f)$, it is still true that $\mathcal{Z}(f) \subset \bigcup_{k=1}^N [\gamma_k]$.

The *zero structure* of quaternion polynomials is a delicate issue (for related results based on various notions of zero multiplicities, we refer to [18], [7], [9]). Here we propose to characterize the left (right) zero structure of a polynomial in terms of its left (right) *spherical divisors*. To introduce these divisors, we first recall several definitions.

The *least right common multiple* $h = \mathbf{lrcm}(f, g)$ of $f, g \in \mathbb{H}[z]$ is defined as a (unique) monic polynomial such that $\langle h \rangle_{\mathbf{r}} = \langle f \rangle_{\mathbf{r}} \cap \langle g \rangle_{\mathbf{r}}$. The least left common multiple $\mathbf{llcm}(f, g)$ is defined as a (unique) monic polynomial generating the left ideal $\langle f \rangle_{\mathbf{l}} \cap \langle g \rangle_{\mathbf{l}}$.

We will say that a (monic) polynomial f is *indecomposable* if it cannot be represented as the least right (left) common multiple of its proper left (right) divisors.

As we will see in Section 5, a polynomial f is indecomposable if and only if it admits a *unique* factorization (1.6) into the product of linear factors and the elements $(\gamma_1, \dots, \gamma_N)$ from this factorization form a spherical chain. In other words, there is a one-to-one correspondence

$$\alpha = (\alpha_1, \dots, \alpha_k) \mapsto P_\alpha := \rho_{\alpha_1} \cdots \rho_{\alpha_k} \quad (1.7)$$

between spherical chains and monic indecomposable polynomials. Here and in what follows we use notation $\rho_\alpha(z) := z - \alpha$ for a fixed $\alpha \in \mathbb{H}$.

Theorem 1.1. *Given $f \in \mathbb{H}[z]$, let V_1, \dots, V_m be distinct conjugacy classes containing zeros of f . Then there exist unique (monic) polynomials D_{ℓ, V_i}^f and $D_{\mathbf{r}, V_i}^f$ ($i = 1, \dots, m$) such that*

$$f = D_{\ell, V_i}^f \cdot h_i \quad \text{so that} \quad \mathcal{Z}(D_{\ell, V_i}^f) \subset V_i, \quad \mathcal{Z}(h_i) \cap V_i = \emptyset, \quad (1.8)$$

$$f = g_i \cdot D_{\mathbf{r}, V_i}^f \quad \text{so that} \quad \mathcal{Z}(D_{\mathbf{r}, V_i}^f) \subset V_i, \quad \mathcal{Z}(g_i) \cap V_i = \emptyset. \quad (1.9)$$

Furthermore, $f = \mathbf{lrcm}(D_{\ell, V_1}^f, \dots, D_{\ell, V_m}^f) = \mathbf{lcm}(D_{\mathbf{r}, V_1}^f, \dots, D_{\mathbf{r}, V_m}^f)$ or equivalently,

$$\langle f \rangle_{\mathbf{r}} = \bigcap_{i=1}^m \langle D_{\ell, V_i}^f \rangle_{\mathbf{r}}, \quad \langle f \rangle_{\ell} = \bigcap_{i=1}^m \langle D_{\mathbf{r}, V_i}^f \rangle_{\ell}. \quad (1.10)$$

Finally, there exist (unique) integers $\kappa_i \geq 0$ and spherical chains $\alpha_i = (\alpha_{i,1}, \dots, \alpha_{i,\kappa_i})$ and $\tilde{\alpha}_i = (\tilde{\alpha}_{i,1}, \dots, \tilde{\alpha}_{i,\kappa_i})$ in V_i such that

$$D_{\ell, V_i}^f = \mathcal{X}_{V_i}^{\kappa_i} \cdot P_{\alpha_i} \quad \text{and} \quad D_{\mathbf{r}, V_i}^f = P_{\tilde{\alpha}_i} \cdot \mathcal{X}_{V_i}^{\kappa_i} \quad \text{for } i = 1, \dots, m, \quad (1.11)$$

where P_{α_i} and $P_{\tilde{\alpha}_i}$ are indecomposable polynomials defined as in (1.7). Consequently,

$$f = \mathcal{X}_{V_1}^{\kappa_1} \cdots \mathcal{X}_{V_m}^{\kappa_m} \cdot \mathbf{lrcm}(P_{\alpha_1}, \dots, P_{\alpha_m}) = \mathcal{X}_{V_1}^{\kappa_1} \cdots \mathcal{X}_{V_m}^{\kappa_m} \cdot \mathbf{lcm}(P_{\tilde{\alpha}_1}, \dots, P_{\tilde{\alpha}_m}). \quad (1.12)$$

We will refer to D_{ℓ, V_j}^f and $D_{\mathbf{r}, V_j}^f$ as to *left and right spherical divisors* of f . Each left (right) spherical divisor contains all information about zeros of f within the corresponding conjugacy class and thus, we may define the left (right) zero structure of f as the collection of its left (right) spherical divisors. The integer κ_i is called the *spherical zero multiplicity* of the conjugacy class V_i . The collections $\{(\kappa_i, \alpha_i)\}_{i=1}^m$ and $\{(\kappa_i, \tilde{\alpha}_i)\}_{i=1}^m$ of spherical multiplicities and spherical chains from representations (1.12) also can serve as definitions of left and right zero structures of a given polynomial f . A result related to Theorem 1.1 is the following version of the Primary Ideal Decomposition Theorem.

Theorem 1.2. *For any non-constant polynomial $f \in \mathbb{H}[z]$, there exist left (right) relatively prime indecomposable polynomials p_1, \dots, p_M (resp., $\tilde{p}_1, \dots, \tilde{p}_M$) such that $f = \mathbf{lrcm}(p_1, \dots, p_M) = \mathbf{lcm}(\tilde{p}_1, \dots, \tilde{p}_M)$. Equivalently,*

$$\langle f \rangle_{\mathbf{r}} = \bigcap_{i=1}^M \langle p_i \rangle_{\mathbf{r}} \quad \text{and} \quad \langle f \rangle_{\ell} = \bigcap_{i=1}^M \langle \tilde{p}_i \rangle_{\ell}. \quad (1.13)$$

In contrast to (1.10), all ideals in (1.13) are irreducible, although the polynomials p_i and \tilde{p}_i in (1.13) are not determined from f uniquely. However, the nonuniqueness can be described explicitly (see Theorem 5.6 below).

The objective of this paper is to establish explicit connections between representations (1.6), (1.10), (1.12), (1.13) and in particular, to construct explicitly a polynomial with the prescribed spherical divisors. One can see from (1.12), that the non-trivial part here is to construct the **lrcm** and **llcm** of m indecomposable polynomials with zeros in m distinct non-real conjugacy classes. The construction has been known for the case where all (but at most one) these polynomials are linear (see e.g., [6]). The algorithm settling the general case is our next contribution to the topic.

The outline of the paper is as follows. After recalling some known results on quaternion polynomials in Section 2, we present the root-finding Algorithm 3.4 (Section 3) and Algorithm 4.3 which produces the spherical divisor of a given polynomial associated with a given conjugacy class. The proof of Theorem 1.1 is given in Section 4.1, and two algorithms relating left and right spherical divisors corresponding to the same conjugacy class are given in Section 4.2. In Section 5, we give several characterizations of indecomposable polynomials and prove Theorem 1.2. The construction of the least common multiple of several indecomposable polynomials is given in Lemma 5.2 (the case where all polynomials have zeros in the same conjugacy class), Algorithm 6.4 (where the polynomials have zeros in distinct conjugacy classes). Finally, Algorithm 6.6 produces a polynomial with prescribed spherical divisors while the least common multiple of given polynomials is the outcome of Algorithm 6.8. In the concluding Section 7, we discuss similar questions in the framework of formal power series over \mathbb{H} .

2. BACKGROUND: QUATERNION POLYNOMIALS AND THEIR ZEROS

A straightforward computation verifies that for any $\alpha \in \mathbb{H}$ and $f \in \mathbb{H}[z]$,

$$f(z) = f^{e\ell}(\alpha) + (z - \alpha) \cdot (L_\alpha f)(z) = f^{er}(\alpha) + (R_\alpha f)(z) \cdot (z - \alpha), \quad (2.1)$$

where $f^{e\ell}(\alpha)$ and $f^{er}(\alpha)$ are respectively, left and right evaluation of f at α given by

$$f^{e\ell}(\alpha) = \sum_{k=0}^m \alpha^k f_k \quad \text{and} \quad f^{er}(\alpha) = \sum_{k=0}^m f_k \alpha^k \quad \text{if} \quad f(z) = \sum_{j=0}^m z^j f_j, \quad (2.2)$$

and where $L_\alpha f$ and $R_\alpha f$ are polynomials of degree $m - 1$ given by

$$(L_\alpha f)(z) = \sum_{k=0}^{m-1} \left(\sum_{j=0}^{m-k-1} \alpha^j f_{k+j+1} \right) z^k, \quad (R_\alpha f)(z) = \sum_{k=0}^{m-1} \left(\sum_{j=0}^{m-k-1} f_{k+j+1} \alpha^j \right) z^k. \quad (2.3)$$

Interpreting $\mathbb{H}[z]$ as a vector space over \mathbb{H} we observe that the mappings $f \mapsto L_\alpha f$ and $f \mapsto R_\alpha f$ define respectively the right linear operator L_α and the left linear operator R_α (called in analogy to the complex case, the left and the right backward shift, respectively) acting on $\mathbb{H}[z]$.

2.1. Left and right zeros. An element $\alpha \in \mathbb{H}$ is called a *left (right) zero* of $f \in \mathbb{H}[z]$ if $f^{e_\ell}(\alpha) = 0$ (respectively, $f^{e_r}(\alpha) = 0$). We denote by $\mathcal{Z}_\ell(f)$ and $\mathcal{Z}_r(f)$ the sets of all left and all right zeros of f , respectively, and we let $\mathcal{Z}(f) := \mathcal{Z}_\ell(f) \cup \mathcal{Z}_r(f)$. The next two equivalences follow from representations (2.1):

$$f^{e_\ell}(\alpha) = 0 \Leftrightarrow f \in \langle \rho_\alpha \rangle_r, \quad f^{e_r}(\alpha) = 0 \Leftrightarrow f \in \langle \rho_\alpha \rangle_\ell. \quad (2.4)$$

Combining equivalences (2.4) gives the following result (see [4, Lemma 2.6] for details).

Lemma 2.1. *Let $\alpha, \beta \in \mathbb{H}$ be two distinct conjugates: $\beta \in [\alpha] \setminus \{\alpha\}$. Then*

$$\langle \rho_\alpha \rangle_r \cap \langle \rho_\beta \rangle_r = \langle \rho_\alpha \rangle_\ell \cap \langle \rho_\beta \rangle_\ell = \langle \mathcal{X}_{[\alpha]} \rangle. \quad (2.5)$$

In case the non-real conjugacy class (the 2-sphere) V is a subset of $\mathcal{Z}_\ell(f) \cap \mathcal{Z}_r(f)$, we will say that V is a *spherical zero* of f .

2.2. Polynomial conjugation. The quaternionic conjugation $\alpha \mapsto \bar{\alpha}$ on \mathbb{H} can be extended to the anti-linear involution $f \mapsto f^\#$ on $\mathbb{H}[z]$ by letting

$$f^\#(z) = \sum_{j=0}^m z^j, \bar{f}_j \quad \text{if} \quad f(z) = \sum_{j=0}^m z^j f_j. \quad (2.6)$$

It is not hard to verify that

$$ff^\# = f^\#f, \quad (fg)^\# = g^\#f^\#, \quad (fg)(fg)^\# = (ff^\#)(gg^\#). \quad (2.7)$$

One can see from (2.2) that if $f \in \mathbb{R}[z]$, then $f^{e_\ell}(\alpha) = f^{e_r}(\alpha)$ for every $\alpha \in \mathbb{H}$; in particular, if $f \in \mathbb{R}[z]$, then $\mathcal{Z}(f) = \mathcal{Z}_\ell(f) = \mathcal{Z}_r(f)$. Moreover, if $f \in \mathbb{R}[z]$, then for each $\alpha \in \mathbb{H}$ and $h \neq 0$, we have $f(h^{-1}\alpha h) = h^{-1}f(\alpha)h$ so that $\mathcal{Z}(f)$ contains, along with each α , the whole conjugacy class $[\alpha]$. Since for any $f \in \mathbb{H}[z]$, the polynomial $ff^\#$ is real, the set $\mathcal{Z}(ff^\#)$ is the union of finitely many conjugacy classes. The following result is essentially due to I. Niven [16].

Theorem 2.2. *Let $\deg(f) \geq 1$ and let $\mathcal{Z}(ff^\#) = \bigcup V_i$ be the union of distinct conjugacy classes. Then $\mathcal{Z}_\ell(f) \cup \mathcal{Z}_r(f) \subset \mathcal{Z}(ff^\#)$ and each conjugacy class V_i either contains exactly one left and one right zero of f or $V_i \in \mathcal{Z}_\ell(f) \cap \mathcal{Z}_r(f)$.*

2.3. Zero multiplicities: If we denote by $f^{(k)}$ the k -th formal derivative of $f \in \mathbb{H}[z]$, then a straightforward verification shows that for any fixed $\alpha \in \mathbb{H}$,

$$f = \sum_{k=0}^{\deg(f)} \rho_\alpha^k \frac{(f^{(k)})^{e_\ell}(\alpha)}{k!} = \sum_{k=0}^{\deg(f)} \frac{(f^{(k)})^{e_r}(\alpha)}{k!} \rho_\alpha^k, \quad \rho_\alpha(z) := z - \alpha. \quad (2.8)$$

Definition 2.3. Let us say that $\alpha \in \mathbb{H}$ is zero of $f \in \mathbb{H}[z]$ of *left zero multiplicity* $m_\ell(\alpha; f) = k$ if $f = \rho_\alpha^k h$ for some $h \in \mathbb{H}[z]$ with $h^{e_\ell}(\alpha) \neq 0$.

It follows from (2.8) that $f \in \langle \rho_\alpha^k \rangle_r$ if and only if $(f^{(j)})^{e_\ell}(\alpha) = 0$ for $j = 0, \dots, k-1$; therefore, $k = m_\ell(\alpha; f)$ can be alternatively defined as the least nonnegative integer such that $(f^{(k)})^{e_\ell}(\alpha) \neq 0$.

The *right zero multiplicity* $m_{\mathbf{r}}(\alpha; f)$ is defined as the integer k such that $f = h\rho_{\alpha}^k$ for some $h \in \mathbb{H}[z]$ with $h^{e_{\mathbf{r}}}(\alpha) \neq 0$, or equivalently, as the least nonnegative integer such that $(f^{(k)})^{e_{\mathbf{r}}}(\alpha) \neq 0$.

Definition 2.4. Let us say that a conjugacy class $V \subset \mathbb{H}$ is the *spherical zero* of $f \in \mathbb{H}[z]$ of the *spherical multiplicity* $\kappa = m_s(V; f)$ if

$$f(z) = \mathcal{X}_V^{\kappa}(z)g(z) = g(z)\mathcal{X}_V^{\kappa}(z) \quad (2.9)$$

for some $g \in \mathbb{H}[z]$ vanishing at at most one point in V . Equivalently, $\kappa = m_s(V; f)$ is the integer such that $f \in \langle \mathcal{X}_V^{\kappa} \rangle \setminus \langle \mathcal{X}_V^{\kappa+1} \rangle$.

The local and spherical zero multiplicities are related as follows: for $f \in \mathbb{H}[z]$ and a conjugacy class $V \subset \mathbb{H}$,

$$m_s(V; f) = \min_{\gamma \in V} \{m_{\ell}(\gamma; f)\} = \min_{\gamma \in V} \{m_{\mathbf{r}}(\gamma; f)\}. \quad (2.10)$$

In fact, as it follows by successive applying Lemma 2.1 to f and its formal derivatives, both minimums in (2.10) are attained at all (but at most one) elements in V .

Remark 2.5. Combining (2.10) with Definition (2.3) of m_{ℓ} and $m_{\mathbf{r}}$ we conclude that for any two elements $\alpha \neq \beta$ in the conjugacy class $V \subset \mathbb{H}$, the spherical zero multiplicity $m_s(V; f)$ equals the least integer $\kappa \geq 0$ such that at least one of the elements $(f^{(\kappa)})^{e_{\ell}}(\alpha)$ and $(f^{(\kappa)})^{e_{\ell}}(\beta)$ (equivalently, one of $(f^{(\kappa)})^{e_{\mathbf{r}}}(\alpha)$ and $(f^{(\kappa)})^{e_{\mathbf{r}}}(\beta)$) is non-zero.

Example 2.6. Let $\alpha \sim \beta \sim \gamma$ be in the same conjugacy class V and let

$$f(z) = (z - \alpha)^n(z - \beta)(z - \gamma)^k.$$

- (1) If $\alpha \neq \bar{\beta} \neq \gamma$, then $m_{\ell}(\alpha; f) = n$, $m_{\mathbf{r}}(\gamma; f) = k$, $m_s(V; f) = 0$.
- (2) In particular, if $\bar{\beta} \neq \alpha = \gamma$, then $m_{\ell}(\alpha; f) = n$, $m_{\mathbf{r}}(\alpha; f) = k$, $m_s(V; f) = 0$.
- (3) If $\gamma = \bar{\beta} \neq \alpha \neq \beta$, then $m_{\ell}(\alpha; f) = n + 1$, $m_{\mathbf{r}}(\gamma; f) = k$, $m_s(V; f) = 1$.
- (4) If $\gamma = \alpha = \bar{\beta}$ and $k \leq n + 1$, then $m_{\ell}(\alpha; f) = m_{\mathbf{r}}(\alpha; f) = n + 1$, $m_s(V; f) = k$.

2.4. Evaluation of products: By (2.4), if $g^{e_{\ell}}(\alpha) = 0$, then $(gf)^{e_{\ell}}(\alpha) = 0$ for any $f \in \mathbb{H}[z]$. On the other hand, since $(gf)(z) = \sum_{k=0}^n z^k g(z) f_k$, we also have

$$(gf)^{e_{\ell}}(\alpha) = g^{e_{\ell}}(\alpha) \sum_{k=0}^n (g^{e_{\ell}}(\alpha)^{-1} \alpha g^{e_{\ell}}(\alpha))^k f_k = g^{e_{\ell}}(\alpha) f^{e_{\ell}}(g^{e_{\ell}}(\alpha)^{-1} \alpha g^{e_{\ell}}(\alpha)),$$

provided $g^{e_{\ell}}(\alpha) \neq 0$. Therefore, the left evaluation of the product of two polynomials is defined by the formula

$$(gf)^{e_{\ell}}(\alpha) = \begin{cases} g^{e_{\ell}}(\alpha) \cdot f^{e_{\ell}}(g^{e_{\ell}}(\alpha)^{-1} \alpha g^{e_{\ell}}(\alpha)) & \text{if } g^{e_{\ell}}(\alpha) \neq 0, \\ 0 & \text{if } g^{e_{\ell}}(\alpha) = 0. \end{cases} \quad (2.11)$$

Similarly, $(gf)^{e_{\mathbf{r}}}(\alpha) = g^{e_{\mathbf{r}}}(f^{e_{\mathbf{r}}}(\alpha) \alpha f^{e_{\mathbf{r}}}(\alpha)^{-1}) \cdot f^{e_{\mathbf{r}}}(\alpha)$ if $f^{e_{\mathbf{r}}}(\alpha) \neq 0$ and $(gf)^{e_{\mathbf{r}}}(\alpha) = 0$ if $f^{e_{\mathbf{r}}}(\alpha) = 0$.

Remark 2.7. Let $\gamma = (\gamma_1, \dots, \gamma_N) \subset [\gamma_1] = V$ be a spherical chain. Then the polynomial $f = \rho_{\gamma_1} \rho_{\gamma_2} \cdots \rho_{\gamma_N}$ has a unique left zero γ_1 and a unique right zero γ_N .

Indeed, since $ff^\sharp = \mathcal{X}_V^N$, it follows by Theorem 2.2 that f has no zeros outside V . On the other hand, since $(\alpha - \beta)^{-1}\beta(\alpha - \beta) = \bar{\alpha}$ for any two distinct $\alpha, \beta \in V$, we conclude from (2.11) that for every $\gamma \in V \setminus \{\gamma_1\}$,

$$f^{e\ell}(\gamma) = (\gamma - \gamma_1)(\bar{\gamma}_1 - \gamma_2)(\bar{\gamma}_2 - \gamma_3) \cdots (\bar{\gamma}_{N-1} - \gamma_N) \neq 0,$$

so that $\mathcal{Z}_\ell(f) = \{\gamma_1\}$. Similarly, one can show that $\mathcal{Z}_r(f) = \{\gamma_N\}$.

3. A “NEW” ROOT-FINDING ALGORITHM FOR QUATERNION POLYNOMIALS

Several known algorithms for finding roots of a quaternion polynomial by means of complex root-finding (see e.g., [16], [20], [13], [14]) assume that all complex zeros of the real polynomial ff^\sharp are known. If x is a real zero of ff^\sharp , then $x \in \mathcal{Z}_\ell(f) \cap \mathcal{Z}_r(f)$, by Theorem 2.2. If $\alpha, \bar{\alpha} \in \mathbb{C}$ are complex-conjugate roots of ff^\sharp and if $f^{e\ell}(\alpha) = f^{e\ell}(\bar{\alpha}) = 0$, then $[\alpha] \subset \mathcal{Z}_\ell(f) \cap \mathcal{Z}_r(f)$, again by Theorem 2.2. The remaining case (finding the only left and the only right zero of f in the conjugacy class $[\alpha]$) is the essence of each individual algorithm. Our contribution here is the following.

Theorem 3.1. Let $f \in \mathbb{H}[z]$, let α and $\bar{\alpha}$ be complex roots (or any quaternion-conjugate roots) of the real polynomial ff^\sharp , and let us assume that $f^{e\ell}(\alpha) \neq 0$. Then the only left root γ_ℓ and the only right root γ_r of f in the conjugacy class $[\alpha]$ are given by

$$\begin{aligned} \gamma_\ell &= (\bar{\alpha}f^{e\ell}(\alpha) + \alpha f^{e\ell}(\bar{\alpha}))(f^{e\ell}(\alpha) + f^{e\ell}(\bar{\alpha}))^{-1}, \\ \gamma_r &= (f^{e\ell}(\alpha) - f^{e\ell}(\bar{\alpha}))^{-1}(\bar{\alpha}f^{e\ell}(\alpha) - \alpha f^{e\ell}(\bar{\alpha})). \end{aligned} \quad (3.1)$$

Proof: If α, β, γ are any three distinct equivalent quaternions, then

$$f^{e\ell}(\gamma) = (\gamma - \beta)(\alpha - \beta)^{-1}f^{e\ell}(\alpha) + (\alpha - \gamma)(\alpha - \beta)^{-1}f^{e\ell}(\beta), \quad (3.2)$$

$$\begin{aligned} f^{er}(\gamma) &= (\alpha - \beta)^{-1}f^{e\ell}(\alpha)\gamma - \beta(\alpha - \beta)^{-1}f^{e\ell}(\alpha) \\ &\quad + \alpha(\alpha - \beta)^{-1}f^{e\ell}(\beta) - (\alpha - \beta)^{-1}f^{e\ell}(\beta)\gamma. \end{aligned} \quad (3.3)$$

Formula (3.2) relating evaluations of the same type was established in [8]; for formula (3.3), we refer to [4, Lemma 3.1]. Letting $\beta = \bar{\alpha}$ simplifies the latter equalities to

$$f^{e\ell}(\gamma) = (\gamma - \bar{\gamma})^{-1}[(\gamma - \bar{\alpha})f^{e\ell}(\alpha) + (\gamma - \alpha)f^{e\ell}(\bar{\alpha})], \quad (3.4)$$

$$f^{er}(\gamma) = (\alpha - \bar{\alpha})^{-1}[f^{e\ell}(\alpha)\gamma - \bar{\alpha}f^{e\ell}(\alpha) + \alpha f^{e\ell}(\bar{\alpha}) - f^{e\ell}(\bar{\alpha})\gamma]. \quad (3.5)$$

Now we will show that $f^{e\ell}(\bar{\alpha}) \neq \pm f^{e\ell}(\alpha)$, so that formulas (3.1) make sense. Indeed, if $f^{e\ell}(\bar{\alpha}) = f^{e\ell}(\alpha)$, then it follows from (3.4) that $f^{e\ell}(\gamma) = f^{e\ell}(\alpha)$ for all $\gamma \in [\alpha]$. On the other hand, if $f^{e\ell}(\bar{\alpha}) = -f^{e\ell}(\alpha)$, then again by (3.4),

$$f^{e\ell}(\gamma) = (\gamma - \bar{\gamma})^{-1}(\alpha - \bar{\alpha})f^{e\ell}(\alpha).$$

Neither of the two latter cases is possible since $f^{e\ell}(\alpha) \neq 0$ by the assumption and since f does have a left root in $[\alpha]$ by Theorem 2.2. Thus $f^{e\ell}(\bar{\alpha}) \neq \pm f^{e\ell}(\alpha)$.

Let γ_ℓ be a left root of f . Then we conclude from (3.4) that

$$(\gamma_\ell - \bar{\alpha})f^{e\ell}(\alpha) + (\gamma_\ell - \alpha)f^{e\ell}(\bar{\alpha}) = 0,$$

and solving the latter equation for γ_ℓ leads us to the first formula in (3.1). Similarly, for the right root γ_r we conclude from (3.5) that

$$f^{e\ell}(\alpha)\gamma_r - \bar{\alpha}f^{e\ell}(\alpha) + \alpha f^{e\ell}(\bar{\alpha}) - f^{e\ell}(\bar{\alpha})\gamma_r = 0,$$

which being solved for γ_r gives the second formula in (3.1). \square

Remark 3.2. Formulas (3.1) for unique left and right roots in a given conjugacy class V are based on two left evaluations of f at two quaternion-conjugate points in V . If $f^{er}(\alpha) \neq 0$, then the formulas for γ_ℓ and γ_r in terms of right evaluations of f are:

$$\begin{aligned}\gamma_\ell &= (f^{er}(\alpha)\bar{\alpha} - f^{er}(\bar{\alpha})\alpha)(f^{er}(\alpha) - f^{er}(\bar{\alpha}))^{-1}, \\ \gamma_r &= (f^{er}(\alpha) + f^{er}(\bar{\alpha}))^{-1}(f^{er}(\alpha)\bar{\alpha} + f^{er}(\bar{\alpha})\alpha).\end{aligned}$$

The proof is similar to that of Theorem 3.1 and will be omitted.

Remark 3.3. If $f^{e\ell}(\bar{\alpha}) = 0$, then formulas (3.1) take the form $\gamma_\ell = \bar{\alpha}$ (as expected) and $\gamma_r = (f^{e\ell}(\alpha))^{-1}\bar{\alpha}f^{e\ell}(\alpha)$. Similarly, if $f^{er}(\bar{\alpha}) = 0$, then formulas in Remark 3.2 give $\gamma_r = \bar{\alpha}$ and $\gamma_\ell = f^{er}(\alpha)\bar{\alpha}(f^{er}(\alpha))^{-1}$.

The root-finding algorithm based on the preceeding analysis follows. Our contribution to the algorithm is part (3d).

Algorithm 3.4. Given a polynomial $f \in \mathbb{H}[z]$ of positive degree,

- (1) Find all complex zeros of the real polynomial ff^\sharp .
- (2) Each real zero of ff^\sharp is a zero of f .
- (3) Evaluate $f^{e\ell}(\alpha)$ and $f^{e\ell}(\bar{\alpha})$ for each pair $\{\alpha, \bar{\alpha}\}$ of complex-conjugate roots ff^\sharp .
 - (a) If $f^{e\ell}(\alpha) = f^{e\ell}(\bar{\alpha}) = 0$, then $[\alpha]$ is the spherical zero of f .
 - (b) If $f^{e\ell}(\alpha) = 0 \neq f^{e\ell}(\bar{\alpha})$, then $\alpha \in \mathcal{Z}_\ell(f)$ and $(f^{e\ell}(\bar{\alpha}))^{-1}\alpha f^{e\ell}(\bar{\alpha}) \in \mathcal{Z}_r(f)$.
 - (c) If $f^{e\ell}(\bar{\alpha}) = 0 \neq f^{e\ell}(\alpha)$, then $\bar{\alpha} \in \mathcal{Z}_\ell(f)$ and $(f^{e\ell}(\alpha))^{-1}\bar{\alpha} f^{e\ell}(\alpha) \in \mathcal{Z}_r(f)$.
 - (d) Otherwise, use formulas (3.1) to compute $\gamma_\ell \in [\alpha] \cap \mathcal{Z}_\ell(f)$ and $\gamma_r \in [\alpha] \cap \mathcal{Z}_r(f)$.

Example 3.5. To illustrate Algorithm 3.4, take $f(z) = z^2 - z(\mathbf{j} + 2\mathbf{k}) + 2\mathbf{i}$. Then

$$(ff^\sharp)(z) = z^4 + 5z^2 + 4 = (z^2 + 1)(z^2 + 4)$$

and therefore, all roots of f are contained in the union of two spheres $V_1 = [\mathbf{i}]$ and $V_2 = [2\mathbf{i}]$. Since $f^{e\ell}(\mathbf{i}) = -1 + 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $f^{e\ell}(-\mathbf{i}) = -1 + 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, the formulas (3.1) give the pair of roots in V_1 :

$$\gamma_\ell = (-2\mathbf{j} - 4\mathbf{k})(-2 + 4\mathbf{i})^{-1} = \mathbf{j}, \quad \gamma_r = (-2\mathbf{k} + 4\mathbf{j})^{-1}(2\mathbf{i} + 4) = 0.8\mathbf{k} - 0.6\mathbf{j}.$$

Similarly, evaluating $f^{e\ell}(\mathbf{i}) = -4 + 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ and $f^{e\ell}(-2\mathbf{i}) = -4 + 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$, we apply formulas (3.1) to $\alpha = 2\mathbf{i}$ to get another pair of roots

$$\gamma_\ell = (-8\mathbf{j} - 16\mathbf{k})(-8 + 4\mathbf{i})^{-1} = 1.6\mathbf{j} + 1.2\mathbf{k}, \quad \gamma_r = (8\mathbf{j} - 4\mathbf{k})^{-1}(8 + 16\mathbf{i}) = 2\mathbf{k}$$

in V_2 . Note that the obtained roots correspond to two different factorizations of f :

$$f(z) = (z - \mathbf{j})(z - 2\mathbf{k}) = (z - (1.6\mathbf{j} + 1.2\mathbf{k}))(z - (0.8\mathbf{k} - 0.6\mathbf{j})).$$

4. FACTORIZATIONS

The standard procedure to factorize a (monic) polynomial $f \in \mathbb{H}[z]$ of degree $N > 0$ into the product N monic linear factors is the following: starting with $Q_0 = f$, construct the sequence $\{\gamma_j\} \subset \mathbb{H}$ and the sequence of monic polynomials $\{Q_j\}$ by

$$\gamma_j \in \mathcal{Z}_\ell(Q_{j-1}), \quad Q_j = L_{\gamma_j} Q_{j-1} \quad \text{for } j = 1, \dots, N, \quad (4.1)$$

where L_α is the left backward shift operator (2.3). Since $\deg(Q_j) = N - j$ and since $Q_{j-1}^{e\ell}(\gamma_j) = 0$, it follows that $Q_{j-1} = \rho_{\gamma_j} Q_j$ and $Q_N \equiv 1$. We now recursively get

$$f = Q_0 = \rho_{\gamma_1} Q_1 = \rho_{\gamma_1} \rho_{\gamma_2} Q_2 = \dots = \rho_{\gamma_1} \rho_{\gamma_2} \dots \rho_{\gamma_N}.$$

The latter factorization is largely non-unique due to non-unique choices of elements γ_j in (4.1). If we will be picking up the elements γ_j from a fixed conjugacy class V for as long as possible, we will recover the zero structure of f within V . Details are furnished in Algorithm 4.1 below. Observe that since ff^\sharp is a real polynomial, $m_s([\alpha]; ff^\sharp) = m_\ell(\alpha; ff^\sharp)$.

Algorithm 4.1. Given $f \in \mathbb{H}[z]$ and a non-real $\alpha \in \mathcal{Z}(ff^\sharp)$, let $k = m_s([\alpha]; ff^\sharp)$.

- (1) Compute recursively $\beta_1, \dots, \beta_k \in \mathbb{H}$ and $Q_1, \dots, Q_k \in \mathbb{H}[z]$ by letting $Q_0 = f$ and

$$\beta_{j+1} = \begin{cases} \alpha, & \text{if } Q_j^{e\ell}(\alpha) = 0, \\ (\overline{\alpha} Q_j^{e\ell}(\alpha) + \alpha Q_j^{e\ell}(\overline{\alpha}))(Q_j^{e\ell}(\alpha) + Q_j^{e\ell}(\overline{\alpha}))^{-1}, & \text{if } Q_j^{e\ell}(\alpha) \neq 0, \end{cases} \quad (4.2)$$

$$Q_{j+1} = L_{\beta_j} Q_j \quad \text{for } j = 0, \dots, k-1, \quad (4.3)$$

where L_{β_j} is the left backward shift operator defined in (2.3). Then

- (1) $\beta_1, \dots, \beta_k \in [\alpha]$; (2) $\mathcal{Z}(Q_k) \cap [\alpha] = \emptyset$; (3) $f = \rho_{\beta_1} \rho_{\beta_2} \dots \rho_{\beta_k} Q_k$.

- (2) Compute recursively $\tilde{\beta}_1, \dots, \tilde{\beta}_k \in \mathbb{H}$ and $\tilde{Q}_1, \dots, \tilde{Q}_k \in \mathbb{H}[z]$ by letting $\tilde{Q}_0 = f$ and

$$\tilde{\beta}_{j+1} = \begin{cases} \alpha, & \text{if } \tilde{Q}_j^{e\ell}(\alpha) = 0, \\ (\tilde{Q}_j^{e\ell}(\alpha) - \tilde{Q}_j^{e\ell}(\overline{\alpha}))^{-1}(\overline{\alpha} \tilde{Q}_j^{e\ell}(\alpha) - \alpha \tilde{Q}_j^{e\ell}(\overline{\alpha})) & \text{if } \tilde{Q}_j^{e\ell}(\alpha) \neq 0, \end{cases} \quad (4.4)$$

$$Q_{j+1} = R_{\tilde{\beta}_j} \tilde{Q}_j \quad \text{for } j = 0, \dots, k-1, \quad (4.5)$$

where $R_{\tilde{\beta}_j}$ is the right backward shift operator defined in (2.3). Then

- (1) $\tilde{\beta}_1, \dots, \tilde{\beta}_k \in [\alpha]$; (2) $\mathcal{Z}(\tilde{Q}_k) \cap [\alpha] = \emptyset$; (3) $f = \tilde{Q}_k \rho_{\tilde{\beta}_k} \dots \rho_{\tilde{\beta}_2} \rho_{\tilde{\beta}_1}$.

Proof: According to Theorem 3.1, the element β_{j+1} defined in (4.2) is a left zero of the polynomial Q_j and belongs to $[\alpha]$. Therefore, for Q_{j+1} defined as in (4.3), we have $Q_j = \rho_{\beta_{j+1}} Q_{j+1}$ from which we recursively recover $f = Q_0 = \rho_{\beta_1} \rho_{\beta_2} \cdots \rho_{\beta_k} Q_k$. By properties (2.7), we conclude from the latter representation that $ff^\sharp = \mathcal{X}_{[\alpha]}^k Q_k Q_k^\sharp$ and since $k = m_s([\alpha]; ff^\sharp)$, it follows that Q_k has no zeros in $[\alpha]$. This completes the proof of the first part of the algorithm. The second part is justified in much the same way. \square

Algorithm 4.1 produces factorizations requested in (1.8), (1.9) (with $V_j = [\alpha]$, $D_{\ell, [\alpha]} = \rho_{\beta_1} \cdots \rho_{\beta_k}$ and $D_{r, [\alpha]} = \rho_{\tilde{\beta}_k} \cdots \rho_{\tilde{\beta}_1}$). As was observed in [9], the polynomial $D_{\ell, [\alpha]}$ (and similarly, $D_{r, [\alpha]}$) can be written in a more structured form if some consecutive points obtained via (4.2) are quaternion-conjugates of each other. If $\beta_{j+1} = \bar{\beta}_j$, then $\rho_{\beta_j} \rho_{\beta_{j+1}} = \rho_{\beta_j} \rho_{\bar{\beta}_j} = \mathcal{X}_{[\alpha]}$, and the latter real polynomial can be commuted through all the factors in $D_{\ell, [\alpha]}$ to the left. Incorporating this observation and Remark 4.2 below, we get a more efficient modification of Algorithm 4.1.

Remark 4.2. For $\alpha \in \mathbb{H}$, a successive application of formulas (2.3) gives

$$R_\alpha R_{\bar{\alpha}} f = L_\alpha L_{\bar{\alpha}} f = \sum_{k=0}^{n-2} z^k \sum_{i=0}^{n-k-2} \left(\sum_{j=0}^i \alpha^j \bar{\alpha}^{i-j} \right) f_{i+k}, \quad \text{if } f(z) = \sum_{k=0}^n z^k f_k. \quad (4.6)$$

Furthermore, if we define the two-terms recursion

$$r_0 = 1, \quad r_1 = 2\operatorname{Re}(\alpha), \quad r_{j+1} = r_j r_1 - r_{j-1} |\alpha|^2 \quad \text{for } j = 1, 2, \dots, \quad (4.7)$$

an inductive argument shows that $r_k = \sum_{j=0}^k \alpha^j \bar{\alpha}^{k-j}$ for all $k \geq 0$. Therefore, the formula for $L_\alpha L_{\bar{\alpha}}$ in (4.6) depends on $\operatorname{Re}(\alpha)$ and $|\alpha|$ rather than α itself and therefore, by characterization (1.3), $L_\alpha L_{\bar{\alpha}} = L_\beta L_{\bar{\beta}}$, whenever $\alpha \sim \beta$. It thus makes sense to introduce the spherical backward shift operator

$$S_{[\alpha]} f = \sum_{k=0}^{n-2} z^k \sum_{i=0}^{n-k-2} r_i f_{i+k} \quad \text{if } f(z) = \sum_{k=0}^n z^k f_k, \quad (4.8)$$

where the real numbers r_i are defined in (4.7).

Algorithm 4.3. Given $f \in \mathbb{H}[z]$ and $\alpha \in \mathbb{H}$, let $m_s([\alpha]; ff^\sharp) = k$.

- (1) Evaluating $f^{(j)}$ at α and $\bar{\alpha}$, find the least integer $\kappa \geq 0$ such that at least one of the elements $(f^{(\kappa)})^{\mathbf{e}_\ell}(\alpha)$ and $(f^{(\kappa)})^{\mathbf{e}_\ell}(\bar{\alpha})$ is non-zero.
- (2) Compute $g = S_{[\alpha]}^\kappa f$ by κ -times application of formula (4.8). If $2\kappa = k$, then g has no zeros in $[\alpha]$ and $f = \mathcal{X}_{[\alpha]}^\kappa g = g \mathcal{X}_{[\alpha]}^\kappa$. Otherwise, proceed to (3a) and (3b).
- (3a) Letting $Q_0 = g$, use recursive formulas (4.2), (4.3) $k - 2\kappa$ times to construct $\alpha_1, \dots, \alpha_{k-2\kappa} \in \mathbb{H}$ and polynomials $Q_1, \dots, Q_{k-2\kappa} = P$.

Then $(\alpha_1, \dots, \alpha_{k-2\kappa}) \subset [\alpha]$ is a spherical chain, P has no zeros in $[\alpha]$, and

$$f = \mathcal{X}_{[\alpha]}^\kappa \rho_{\alpha_1} \rho_{\alpha_2} \cdots \rho_{\alpha_{k-2\kappa}} P. \quad (4.9)$$

(3b) Letting $\tilde{Q}_0 = g$, use recursive formulas (4.4), (4.5) $k - 2\kappa$ times to construct $\tilde{\alpha}_1, \dots, \tilde{\alpha}_{k-2\kappa}$ and polynomials $\tilde{Q}_1, \dots, \tilde{Q}_{k-2\kappa} = \tilde{P}$.

Then $(\tilde{\alpha}_1, \dots, \tilde{\alpha}_{k-2\kappa}) \subset [\alpha]$ is a spherical chain, \tilde{P} has no zeros in $[\alpha]$, and

$$f = \tilde{P} \rho_{\tilde{\alpha}_{k-2\kappa}} \cdots \rho_{\tilde{\alpha}_2} \rho_{\tilde{\alpha}_1} \mathcal{X}_{[\alpha]}^\kappa. \quad (4.10)$$

Proof: The integer κ obtained in Step (1) equals $m_s([\alpha]; f)$ by Remark 2.5. Hence the polynomial $g = S_{[\alpha]}^\kappa f$ satisfies (2.9), and $m_s([\alpha]; g) = 0$. On the other hand, since $ff^\# = \mathcal{X}_{[\alpha]}^{2\kappa} gg^\#$ (by (2.7)), it follows that $m_s([\alpha]; gg^\#) = k - 2\kappa$ and we may apply Algorithm 4.1 (part 1) to the polynomial g to get $\alpha_1, \dots, \alpha_{k-2\kappa} \in [\alpha]$ and the polynomial $P_{k-2\kappa}$ with no zeros in $[\alpha]$ such that $g = \rho_{\alpha_1} \rho_{\alpha_2} \cdots \rho_{\alpha_{k-2\kappa}} P_{k-2\kappa}$. If $\alpha_{j+1} = \bar{\alpha}_j$ for some j , then $\rho_{\alpha_j} \rho_{\alpha_{j+1}} = \mathcal{X}_{[\alpha]}$ so that $m_s([\alpha]; g) \geq 1$ which is a contradiction. Therefore $\alpha_{j+1} \neq \bar{\alpha}_j$ for all j as desired. Applying the second part of Algorithm 4.1 to the polynomial g , we get representation (4.10). \square

Remark 4.4. The representation (4.9) is unique in the following sense: if

$$f = \mathcal{X}_{[\alpha]}^{\kappa'} \rho_{\alpha'_1} \rho_{\alpha'_2} \cdots \rho_{\alpha'_s} G, \quad \mathcal{Z}_\ell(G) \cap [\alpha] = \emptyset, \quad (4.11)$$

is another factorization of f with the spherical chain $(\alpha'_1, \dots, \alpha'_s) \in [\alpha]$, then $\kappa' = \kappa$, $s = k - 2\kappa$, $\alpha'_j = \alpha_j$ for $j = 1, \dots, s$, and $G = P$. The representation (4.10) is unique in a similar sense.

Proof: The integers κ and κ' are both equal to $m_s([\alpha], f)$ and therefore, $\kappa' = \kappa$. Then we have from (4.9) and (4.11),

$$g = \rho_{\alpha_1} \rho_{\alpha_2} \cdots \rho_{\alpha_{k-2\kappa}} P_{k-2\kappa} = \rho_{\alpha'_1} \rho_{\alpha'_2} \cdots \rho_{\alpha'_s} G. \quad (4.12)$$

Therefore $gg^\# = \mathcal{X}_{[\alpha]}^{2k-4\kappa} P_{k-2\kappa} P_{k-2\kappa}^\# = \mathcal{X}_{[\alpha]}^{2s} G G^\#$ and since $P_{k-2\kappa}$ and G have no zeros in $[\alpha]$, we conclude that $k - 2\kappa = s$. By Remark 2.7, it follows from factorizations (4.12) that α_1 (and also α'_1) is a unique left zero of g in V . Therefore, $\alpha_1 = \alpha'_1$ and applying L_{α_1} to equalities (4.12) gives

$$L_{\alpha_1} g = \rho_{\alpha_2} \cdots \rho_{\alpha_s} P = \rho_{\alpha'_2} \cdots \rho_{\alpha'_s} G.$$

Repeating the above argument we subsequently conclude that $\alpha_j = \alpha'_j$ for all $j = 1, \dots, s$ and then also $P = G$. \square

Corollary 4.5. Any monic polynomial f with all zeros contained in the non-real conjugacy class V can be (uniquely) factored either as $f = \mathcal{X}_{[\alpha]}^\kappa$ or as

$$f = \mathcal{X}_V^\kappa \rho_{\alpha_1} \cdots \rho_{\alpha_n} \quad (\alpha_j \in V, \alpha_{j+1} \neq \bar{\alpha}_j). \quad (4.13)$$

The statement follows from representation (4.9) and Remark 4.3, since the monic polynomial P in (4.9) does not have roots and therefore $P \equiv 1$.

Example 4.6. To illustrate Algorithm 4.3, let us consider the polynomial

$$\begin{aligned} f(z) = & z^7 - (1 + \mathbf{i} + \mathbf{j} + \mathbf{k})z^6 + (2 - \mathbf{i} + 2\mathbf{j})z^5 - (3 + \mathbf{i} + 2\mathbf{j} + 2\mathbf{k})z^4 \\ & + (1 - 2\mathbf{i} + 4\mathbf{j})z^3 - (3 - \mathbf{i} + \mathbf{j} + \mathbf{k})z^2 + (2\mathbf{j} - \mathbf{i})z + \mathbf{i} - 1. \end{aligned}$$

A straightforward computation shows that

$$(ff^\sharp)(z) = z^{14} - 2z^{13} + 8z^{11} + 27z^{10} - 30z^9 + 50z^8 - 40z^7 + 55z^6 - 30z^5 \\ + 36z^4 - 12z^3 + 13z^2 - 2z + 2 = (z^2 + 1)^6(z^2 - 2z + 2).$$

We see that all zeros of f are contained in the conjugacy classes $V_1 = [\mathbf{i}]$ and $V_2 = [1 + \mathbf{i}]$. V_2 contains isolated zeros of multiplicity one, whereas V_1 contains zeros of higher multiplicities. Applying Algorithm 4.3 we first evaluate: $f^{e\ell}(\mathbf{i}) = f^{e\ell}(-\mathbf{i}) = 0$, $(f')^{e\ell}(\mathbf{i}) = (f')^{e\ell}(-\mathbf{i}) = 0$ and

$$(f'')^{e\ell}(\mathbf{i}) = 42\mathbf{i}^5 - 30\mathbf{i}^4(1 + \mathbf{i} + \mathbf{j} + \mathbf{k}) + 20\mathbf{i}^3(2 - \mathbf{i} + 2\mathbf{j}) - 12\mathbf{i}^2(3 + \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) \\ + 6\mathbf{i}(1 - 2\mathbf{i} + 4\mathbf{j}) - 2(3 - \mathbf{i} + \mathbf{j} + \mathbf{k}) = -8 - 8\mathbf{i} - 8\mathbf{j} - 24\mathbf{k} \neq 0.$$

Thus, $m_s([\mathbf{i}], f) = 2$. The recursion (4.7) takes the form

$$r_0 = 1, \quad r_i = 0, \quad r_{j+1} = -r_{j-1} \quad \text{for } j = 1, 2, \dots$$

and we subsequently get

$$S_{[\mathbf{i}]}f = z^5 - (1 + \mathbf{i} + \mathbf{j} + \mathbf{k})z^4 + (1 - \mathbf{i} + 2\mathbf{j})z^3 - (2 + \mathbf{j} + \mathbf{k})z^2 + (2\mathbf{j} - \mathbf{i})z + \mathbf{i} - 1, \\ g := S_{[\mathbf{i}]}^2f = z^3 - (1 + \mathbf{i} + \mathbf{j} + \mathbf{k})z^2 - (\mathbf{i} - 2\mathbf{j})z + \mathbf{i} - 1.$$

Since $m_s([\mathbf{i}], ff^\sharp) = 4$ and $m_s([\mathbf{i}], f) = 2$, we proceed to Step 3. We have

$$g^{e\ell}(\mathbf{i}) = 1 + \mathbf{i} + \mathbf{j} + 3\mathbf{k}, \quad g^{e\ell}(-\mathbf{i}) = -1 + 3\mathbf{i} + \mathbf{j} - \mathbf{k}, \quad (4.14)$$

and therefore, by part (3a) of Algorithm 4.3,

$$\alpha_1 = (-\mathbf{i}g^{e\ell}(\mathbf{i}) + \mathbf{i}g^{e\ell}(-\mathbf{i}))(g^{e\ell}(\mathbf{i}) + g^{e\ell}(-\mathbf{i}))^{-1} = \mathbf{k}, \\ Q_1 = L_{\alpha_1}g = L_{\mathbf{k}}g = z^2 - (1 + \mathbf{i} + \mathbf{j})z + \mathbf{j} - \mathbf{k}.$$

We next compute $Q_1^{e\ell}(\mathbf{i}) = -\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $Q_1^{e\ell}(-\mathbf{i}) = -2 + \mathbf{i} + \mathbf{j}$ and subsequently get

$$\alpha_2 = (-\mathbf{i}Q_1^{e\ell}(\mathbf{i}) + \mathbf{i}Q_1^{e\ell}(-\mathbf{i}))(Q_1^{e\ell}(\mathbf{i}) + Q_1^{e\ell}(-\mathbf{i}))^{-1} = \mathbf{j}, \\ Q_2 = L_{\alpha_2}Q_1 = L_{\mathbf{j}}Q_1 = z - 1 - \mathbf{i}.$$

The representation (4.9) for f takes the form

$$f(z) = (z^2 + 1)^2(z - \mathbf{k})(z - \mathbf{j})P(z), \quad \text{where } P(z) = (z - 1 - \mathbf{i}). \quad (4.15)$$

Applying part (3b) of Algorithm 4.3 and making use of (4.14), we get

$$\tilde{\alpha}_1 = (g^{e\ell}(\mathbf{i}) - g^{e\ell}(-\mathbf{i}))^{-1}(-\mathbf{i}g^{e\ell}(\mathbf{i}) - \mathbf{i}g^{e\ell}(-\mathbf{i})) = \frac{2\mathbf{i} + \mathbf{j} - 2\mathbf{k}}{3}, \\ \tilde{Q}_1 = R_{\tilde{\alpha}_1}g = z^2 - \frac{3 + \mathbf{i} + 2\mathbf{j} + 5\mathbf{k}}{3}z + \frac{-2 - 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}}{3}.$$

We then compute

$$\tilde{Q}_1^{e\ell}(\mathbf{i}) = \frac{-4 - 5\mathbf{i} + 6\mathbf{j} + \mathbf{k}}{3}, \quad \tilde{Q}_1^{e\ell}(-\mathbf{i}) = \frac{-6 + \mathbf{i} - 4\mathbf{j} + 5\mathbf{k}}{3}, \\ \tilde{\alpha}_2 = (\tilde{Q}_1^{e\ell}(\mathbf{i}) - \tilde{Q}_1^{e\ell}(-\mathbf{i}))^{-1}(-\mathbf{i}\tilde{Q}_1^{e\ell}(\mathbf{i}) - \mathbf{i}\tilde{Q}_1^{e\ell}(-\mathbf{i})) = \frac{-2\mathbf{i} + 26\mathbf{j} + 29\mathbf{k}}{39}, \\ \tilde{Q}_2 = R_{\tilde{\alpha}_2}\tilde{Q}_1 = z - 1 + \frac{5\mathbf{i} + 12\mathbf{k}}{13},$$

and representation (4.10) for f takes the form

$$f(z) = \tilde{P}(z) \left(z - \frac{-2\mathbf{i} + 26\mathbf{j} + 29\mathbf{k}}{39} \right) \left(z - \frac{2\mathbf{i} + \mathbf{j} - 2\mathbf{k}}{3} \right) (z^2 + 1)^2, \quad (4.16)$$

where $\tilde{P}(z) = z - 1 - \frac{5\mathbf{i} + 12\mathbf{k}}{13}$.

4.1. Proof of Theorem 1.1. Let V be a conjugacy class containing zeros of a given $f \in \mathbb{H}[z]$. If $V = \{x\}$ where x is a real root of f of multiplicity k , then $f = \rho_x^k h = h \rho_x^k$ are the factorizations requested in (1.8), (1.9). In the non-real case, representations (1.8), (1.9) are established by Algorithm 4.3 with

$$D_{\ell, [\alpha]}^f = \mathcal{X}_{[\alpha]}^\kappa \rho_{\alpha_1} \cdots \rho_{\alpha_{k-2\kappa}} \quad \text{and} \quad D_{\mathbf{r}, [\alpha]}^f = \mathcal{X}_{[\alpha]}^\kappa \rho_{\tilde{\alpha}_{k-2\kappa}} \cdots \rho_{\tilde{\alpha}_1}, \quad (4.17)$$

proving, therefore, also formulas (1.11). The uniqueness of factorizations (1.8), (1.9) and (1.11) was shown in Remark 4.3 and Corollary 4.5. Since the real polynomial ff^\sharp has only spherical zeros or isolated real zeros of even multiplicities, it can be factored as $ff^\sharp = \prod_{j=1}^m \mathcal{X}_{V_j}^{k_j}$, where we again let $\mathcal{X}_{[x]} = \rho_x^2$ if $x \in \mathbb{R}$. It follows from Algorithm 4.3 and formulas (4.17) that $\deg(D_{\ell, V_j}^f) = k_j$ so that $\deg f = \sum_{j=1}^m k_j$. Observe that for any right common multiple F of $D_{\ell, V_1}^f, \dots, D_{\ell, V_m}^f$, the polynomial FF^\sharp is a common multiple of relatively prime real polynomials $D_{\ell, V_j}^f D_{\ell, V_j}^\sharp = \mathcal{X}_{V_j}^{k_j}$ ($1 \leq j \leq m$) and therefore, $\deg F \geq \sum_{j=1}^m k_j$. Thus, f is a right common multiple of $D_{\ell, V_1}^f, \dots, D_{\ell, V_m}^f$ (by (1.8)) of the minimally possible degree. Therefore, $f = \text{lrcm}(D_{\ell, V_1}^f, \dots, D_{\ell, V_m}^f)$ which proves the first equality in (1.10). The second equality follows similarly. Equalities (1.12) now follow from (1.11). \square

4.2. The right zero structure versus the left: If $f \in \mathbb{H}[z]$ is completely factored as in (1.6), we can construct its spherical divisors using Algorithm 4.3. If f is given in the form (4.9), then its left zero structure is known only within the conjugacy class $[\alpha]$. However, this information is sufficient to recover the right zero structure of f within $[\alpha]$. We recall the backward shift operators L_α and R_β defined in (2.3).

Lemma 4.7. (1) If $F \in \mathbb{H}[z]$ and $\gamma \in \mathbb{H}$ are such that $\mathcal{Z}(F) \cap [\gamma] = \emptyset$, then

$$\rho_\gamma F = Q \rho_\beta, \quad \text{where} \quad \beta = (F^{e_\ell}(\bar{\gamma}))^{-1} \gamma F^{e_\ell}(\bar{\gamma}), \quad Q = R_\beta(\rho_\gamma F).$$

Moreover, $\mathcal{Z}(Q) \cap [\gamma] = \emptyset$ and $\gamma = F^{e_r}(\beta) \beta (F^{e_r}(\beta))^{-1}$.

(2) If $Q \in \mathbb{H}[z]$ and $\beta \in \mathbb{H}$ are such that $\mathcal{Z}(Q) \cap [\beta] = \emptyset$, then

$$Q \rho_\beta = \rho_\gamma F, \quad \text{where} \quad \gamma = Q^{e_r}(\bar{\beta}) \beta (Q^{e_r}(\bar{\beta}))^{-1}, \quad F = L_\gamma(Q \rho_\beta).$$

Moreover, $\mathcal{Z}(F) \cap [\beta] = \emptyset$ and $\beta = (Q^{e_\ell}(\gamma))^{-1} \gamma Q^{e_\ell}(\gamma)$.

Proof: The polynomial $g = \rho_\gamma F$ has a unique left zero in $[\gamma]$ (which is γ). By Remark 3.3, the unique right zero of g in $[\gamma]$ is given by

$$\beta = g^{e_\ell}(\bar{\gamma})^{-1} \gamma g^{e_\ell}(\bar{\gamma}) = ((\bar{\gamma} - \gamma) F^{e_\ell}(\bar{\gamma}))^{-1} \gamma (\bar{\gamma} - \gamma) F^{e_\ell}(\bar{\gamma}) = (F^{e_\ell}(\bar{\gamma}))^{-1} \gamma F^{e_\ell}(\bar{\gamma}).$$

By (2.1), g can be factored as $g = Q\rho_\beta$, where $Q = R_\beta g$. Since β is the only right zero of g in $[\gamma]$, it follows that $\mathcal{Z}(Q) \cap [\gamma] = \emptyset$. Finally, evaluating both parts in $\rho_\gamma F = Q\rho_\beta$ at $z = \beta$ on the right gives

$$F^{er}(\beta)\beta - \gamma F^{er}(\beta) = 0$$

from which we conclude $\gamma = F^{er}(\beta)\beta(F^{er}(\beta))^{-1}$. This completes the proof of the first statement of the lemma. The second statement is verified in much the same way. \square

The next algorithm recovers the right spherical divisor of a given polynomial from the given left spherical divisor associated with the same conjugacy class.

Algorithm 4.8. *Given $f \in \mathbb{H}[z]$ in the form*

$$f = D_{\ell, [\alpha]}^f P, \quad \text{where} \quad D_{\ell, [\alpha]}^f = \mathcal{X}_{[\alpha]}^\kappa \rho_{\alpha_1} \rho_{\alpha_2} \cdots \rho_{\alpha_n} \quad (\alpha_{j+1} \neq \bar{\alpha}_j), \quad (4.18)$$

let $P_0 = P$ and recursively compute

$$\tilde{\alpha}_{j+1} = P_j^{e\ell}(\bar{\alpha}_{n-j})^{-1} \alpha_{n-j} P_j^{e\ell}(\bar{\alpha}_{n-j}), \quad P_{j+1} = R_{\tilde{\alpha}_{j+1}}(\rho_{\alpha_{n-j}} P_j) \quad (4.19)$$

for $j = 0, \dots, n-1$. Then f can be represented as

$$f = \tilde{P} D_{\mathbf{r}, [\alpha]}^f, \quad \text{where} \quad D_{\mathbf{r}, [\alpha]}^f = \rho_{\tilde{\alpha}_n} \cdots \rho_{\tilde{\alpha}_2} \rho_{\tilde{\alpha}_1} \mathcal{X}_{[\alpha]}^\kappa, \quad \tilde{P} = P_n. \quad (4.20)$$

Proof: Based on the first statement in Lemma 4.7 and definitions (4.19), a simple induction argument shows that

$$\rho_{\alpha_{n-j}} P_j = P_{j+1} \rho_{\tilde{\alpha}_j} \quad \text{and} \quad \mathcal{Z}(P_{j+1}) \cap [\alpha] = \emptyset \quad \text{for} \quad j = 0, \dots, n-1.$$

Therefore, we get recursively

$$\begin{aligned} \rho_{\alpha_1} \cdots \rho_{\alpha_{n-1}} \rho_{\alpha_n} P &= \rho_{\alpha_1} \cdots \rho_{\alpha_{n-1}} \rho_{\alpha_n} P_0 = \rho_{\alpha_1} \cdots \rho_{\alpha_{n-1}} P_1 \rho_{\tilde{\alpha}_1} \\ &= \rho_{\alpha_1} \cdots \rho_{\alpha_{n-2}} P_2 \rho_{\tilde{\alpha}_2} \rho_{\tilde{\alpha}_1} = \dots = P_n \rho_{\tilde{\alpha}_n} \rho_{\tilde{\alpha}_{n-1}} \cdots \rho_{\tilde{\alpha}_1}. \end{aligned}$$

Multiplying the last equality by $\mathcal{X}_{[\alpha]}^\kappa$ (from either side, since $\mathcal{X}_{[\alpha]} \in \mathbb{R}[z]$) and taking into account (4.18) we get (4.20). Since $m_s([\alpha; f]) = \kappa$ by (4.18), it also follows that $\tilde{\alpha}_{j+1} \neq \tilde{\alpha}_j$ for $j = 0, \dots, n-1$. \square

Example 4.9. *Let us consider the polynomial f from Example 4.6 and its factorization (4.15). Thus, $\alpha_1 = \mathbf{k}$, $\alpha_2 = \mathbf{j}$ and $P(z) = z - 1 - \mathbf{i}$. We now apply Algorithm*

4.8:

$$\begin{aligned}
\tilde{\alpha}_1 &= (-\mathbf{j} - 1 - \mathbf{i})^{-1} \mathbf{j} (-\mathbf{j} - 1 - \mathbf{i}) = \frac{2\mathbf{i} + \mathbf{j} - 2\mathbf{k}}{3}, \\
\rho_{\alpha_2} P_0 &= z^2 - (1 + \mathbf{i} + \mathbf{j})z + \mathbf{j} - \mathbf{k}, \quad P_1 = R_{\tilde{\alpha}_1}(\rho_{\alpha_2} P_0) = z - 1 - \frac{\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}}{3}, \\
\tilde{\alpha}_2 &= \left(-\mathbf{k} - 1 - \frac{\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}}{3} \right)^{-1} \mathbf{k} \left(-\mathbf{k} - 1 - \frac{\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}}{3} \right) = \frac{-2\mathbf{i} + 26\mathbf{j} + 29\mathbf{k}}{39}, \\
\rho_{\alpha_1} P_1 &= z^2 - \frac{3 + \mathbf{i} + 2\mathbf{j} + 5\mathbf{k}}{3}z + \frac{-2 - 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}}{3}, \\
P_2 &= R_{\tilde{\alpha}_2}(\rho_{\alpha_1} P_1) = z - 1 - \frac{5\mathbf{i} + 12\mathbf{k}}{13},
\end{aligned}$$

and the factorization $f = P_2 \rho_{\tilde{\alpha}_2} \rho_{\tilde{\alpha}_1} \mathcal{X}_{[\mathbf{i}]}^2$ coincides with that in (4.16), as expected.

We conclude the section with the algorithm recovering $D_{\ell, [\alpha]}^f$ from $D_{\mathbf{r}, [\alpha]}^f$; justification is based on the second statement in Lemma 4.7 and will be omitted.

Algorithm 4.10. Given $f \in \mathbb{H}[z]$ of the form (4.20), let $P_0 = P$ and recursively compute

$$\alpha_{j+1} = P_j^{er}(\tilde{\alpha}_{n-j})^{-1} \tilde{\alpha}_{n-j} P_j^{er}(\tilde{\alpha}_{n-j}), \quad P_{j+1} = L_{\alpha_{j+1}}(P_j \rho_{\tilde{\alpha}_{n-j}}) \quad (4.21)$$

for $j = 0, \dots, n-1$. Then f can be represented as in (4.18) with $P = P_n$.

5. INDECOMPOSABLE POLYNOMIALS AND IRREDUCIBLE DECOMPOSITIONS

Let us say that a left (right) ideal in $\mathbb{H}[z]$ is irreducible if it is not contained properly in two distinct ideals of the same type. The generators of irreducible ideals, therefore, are the polynomials that cannot be represented as the least right (left) common multiple of their proper left (right) divisors. In [17], such polynomials were called *indecomposable*. In the next theorem, we collect a number of equivalent characterizations of indecomposable polynomials.

Theorem 5.1. Let $f \in \mathbb{H}[z]$ be a monic polynomial factored as in (1.6):

$$f(z) = (z - \gamma_1)(z - \gamma_2) \cdots (z - \gamma_N), \quad \gamma_1, \dots, \gamma_N \in \mathbb{H}. \quad (5.1)$$

The following are equivalent:

- (1) $\gamma = (\gamma_1, \dots, \gamma_N)$ is a spherical chain.
- (2) γ_1 is the only left zero of f .
- (3) γ_N is the only right zero of f .
- (4) (5.1) is a unique factorization of f into the product of linear factors.
- (5) The ideal $\langle f \rangle_{\mathbf{r}}$ is irreducible.
- (6) The ideal $\langle f \rangle_{\ell}$ is irreducible.

Proof: The implication (1) \Rightarrow (2) was verified in Remark 2.7. Let us assume that (1) is not in force, i.e., that either $\gamma_1 \not\sim \gamma_j$ or $\gamma_{j+1} = \bar{\gamma}_j$ for some $j \in \{1, \dots, N\}$. In the first case, the conjugacy class $[\gamma_j]$ contains a left zero α of f different from γ_1 ;

in the second case, $\rho_{\gamma_j} \rho_{\gamma_{j+1}} = \rho_{\gamma_j} \rho_{\overline{\gamma}_j} = \mathcal{X}_V$ so that $f \in \langle \mathcal{X}_V \rangle$ and therefore f has infinitely many left zeros. This completes the proof of $(1) \Leftrightarrow (2)$.

Let us assume that (1) holds and let $f(z) = (z - \gamma'_1)(z - \gamma'_2) \cdots (z - \gamma'_N)$ be another factorization of f into the product of linear factors. Since both γ_1 and γ'_1 are left zeros of f and since γ_1 is the only left zero of f by $(1) \Rightarrow (2)$, it follows that $\gamma_1 = \gamma'_1$. We then consider the equality

$$L_{\gamma_1} f = \rho_{\gamma_2} \cdots \rho_{\gamma_N} = \rho_{\gamma'_2} \cdots \rho_{\gamma'_N}$$

and since still $\gamma_{j+1} \neq \overline{\gamma}_j$ for $j = 2, \dots, N-1$, we conclude as above that γ_2 is the only left zero of the polynomial $L_{\gamma_1} f$ and that $\gamma_2 = \gamma'_2$. We subsequently get $\gamma_j = \gamma'_j$ so that (5.1) is indeed a unique factorization of f . This completes the proof of $(1) \Rightarrow (4)$.

To prove $(4) \Rightarrow (5)$, observe that the uniqueness of (5.1) implies that any left divisor h of f is of the form $h(z) = (z - \gamma_1)(z - \gamma_2) \cdots (z - \gamma_n)$ for some $n \leq N$ (to see this, it suffices to compare factorization $f = hg$ with (5.1)). Therefore, for any two proper left divisors $h = \rho_{\gamma_1} \cdots \rho_{\gamma_n}$ and $\tilde{h} = \rho_{\gamma_1} \cdots \rho_{\gamma_k}$ ($n \leq k < N$) of f , we have $\langle h \rangle_{\mathbf{r}} \cap \langle \tilde{h} \rangle_{\mathbf{r}} = \langle \tilde{h} \rangle_{\mathbf{r}} \neq \langle f \rangle_{\mathbf{r}}$. Therefore, the ideal $\langle f \rangle_{\mathbf{r}}$ is irreducible.

To prove $(5) \Rightarrow (2)$, let us assume that the ideal $\langle f \rangle_{\mathbf{r}}$ is irreducible. If f has zeros in more than one conjugacy class, then f is equal to the **lrcm** of its left spherical divisors, by Theorem 1.1. Since in this case, each left spherical divisor of f is a proper left divisor of f , it follows that the ideal $\langle f \rangle_{\mathbf{r}}$ is not irreducible which is a contradiction. Therefore $\mathcal{Z}(f) \subset [\alpha]$ for some $\alpha \in \mathbb{H} \setminus \mathbb{R}$. If $f = \mathcal{X}_{[\alpha]}^\kappa$ for $\kappa \geq 1$, then the polynomials $g = \rho_\alpha^\kappa$ and $h = \rho_{\overline{\alpha}}^\kappa$ are proper left divisors of f and their **lrcm** equals f . Therefore, $\langle f \rangle_{\mathbf{r}}$ is not irreducible which contradicts the current assumption. It now follows from Corollary 4.5 that f is necessarily of the form (4.13). If $\kappa = m_s([\alpha]; f) > 0$, then the polynomials

$$g = \rho_{\alpha_1} \cdots \rho_{\alpha_{n-1}} \rho_{\alpha_n}^{\kappa+1} \quad \text{and} \quad h = \rho_{\overline{\alpha}_1}^\kappa$$

are proper left divisors of f and their least right common multiple equals f (the details are furnished in Lemma 5.4 below; see also Remark 5.5). We again conclude that $\langle f \rangle_{\mathbf{r}}$ is not irreducible which contradicts the current assumption. Therefore, $\kappa = 0$ in representation (4.13) and f has a unique left zero.

We have verified implications $(1) \Leftrightarrow (2) \Rightarrow (4) \Rightarrow (5) \Rightarrow (2)$. Implications $(1) \Leftrightarrow (3) \Rightarrow (4) \Rightarrow (6) \Rightarrow (3)$ are verified in much the same way. \square

Notation: In what follows, we will write $\mathcal{P}_V := \{f \in \mathbb{H}[z] : \mathcal{Z}(f) \subset V\}$ for the set of polynomials having all zeros in V , and we will denote by \mathcal{IP}_V the set of (indecomposable) polynomials having one left and one right zero in V .

Lemma 5.2. *Let $g, h \in \mathcal{IP}_V$ ($\deg(g) = n \geq k = \deg(h)$) be given in the form*

$$g = \rho_{\alpha_1} \cdots \rho_{\alpha_n}, \quad h = \rho_{\beta_1} \cdots \rho_{\beta_k} \tag{5.2}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_k)$ are two spherical chains from the conjugacy class V . If g and h are left coprime (i.e., if $\alpha_1 \neq \beta_1$), then

$$f := \text{lrcm}(g, h) = \begin{cases} \mathcal{X}_V^k, & \text{if } n = k, \\ \mathcal{X}_V^k \rho_{\alpha_1} \rho_{\alpha_2} \cdots \rho_{\alpha_{n-k}}, & \text{if } n > k. \end{cases} \quad (5.3)$$

Similarly, if g and h are right coprime (i.e., if $\alpha_n \neq \beta_k$), then

$$\tilde{f} := \text{llcm}(g, h) = \begin{cases} \mathcal{X}_V^k, & \text{if } n = k, \\ \mathcal{X}_V^k \rho_{\alpha_{k+1}} \rho_{\alpha_{k+2}} \cdots \rho_{\alpha_n}, & \text{if } n > k. \end{cases} \quad (5.4)$$

Proof: Since f is a right common multiple of g and h we have from (5.2)

$$f = \rho_{\alpha_1} \cdots \rho_{\alpha_{n-1}} \rho_{\alpha_n} p = \rho_{\beta_1} \cdots \rho_{\beta_{k-1}} \rho_{\beta_k} q \quad \text{for some } p, q \in \mathbb{H}[z]. \quad (5.5)$$

We then use (2.11) to evaluate the latter representations at α_1 and at β_1 on the left:

$$\begin{aligned} f^{e\ell}(\beta_1) &= (\beta_1 - \alpha_1)(\bar{\alpha}_1 - \alpha_2) \cdots (\bar{\alpha}_{n-1} - \alpha_n) p^{e\ell}(\bar{\alpha}_n) = 0, \\ f^{e\ell}(\alpha_1) &= 0 = (\alpha_1 - \beta_1)(\bar{\beta}_1 - \beta_2) \cdots (\bar{\beta}_{k-1} - \beta_k) q^{e\ell}(\bar{\beta}_k). \end{aligned}$$

Since $\alpha_1 \neq \beta_1$, $\alpha_{j+1} \neq \bar{\alpha}_j$ and $\beta_{j+1} \neq \bar{\beta}_j$, the latter equalities imply $p^{e\ell}(\bar{\alpha}_n) = q^{e\ell}(\bar{\beta}_k) = 0$. By (2.4), $p = \rho_{\bar{\alpha}_n} p_1$ and $q = \rho_{\bar{\beta}_k} q_1$ for some $p_1, q_1 \in \mathbb{H}[z]$. Substituting the latter factorizations into (5.5) gives

$$\begin{aligned} f &= \rho_{\alpha_1} \cdots \rho_{\alpha_{n-1}} \rho_{\alpha_n} \rho_{\bar{\alpha}_n} p_1 = \rho_{\alpha_1} \cdots \rho_{\alpha_{n-1}} p_1 \mathcal{X}_V \\ &= \rho_{\beta_1} \cdots \rho_{\beta_{k-1}} \rho_{\beta_k} \rho_{\bar{\beta}_k} q_1 = \rho_{\beta_1} \cdots \rho_{\beta_{k-1}} q_1 \mathcal{X}_V. \end{aligned}$$

Applying the spherical backward shift S_V (4.8) to the latter equalities gives

$$S_V f = \rho_{\alpha_1} \cdots \rho_{\alpha_{n-1}} p_1 = \rho_{\beta_1} \cdots \rho_{\beta_{k-1}} q_1.$$

Repeating the preceding argument $k-1$ more times we get polynomials p_1, \dots, p_k and q_1, \dots, q_k such that $p_j = \rho_{\alpha_{n-j}} p_{j+1}$ and $q_j = \rho_{\alpha_{k-j}} q_{j+1}$, and eventual equalities

$$S_V^k f = p_k = q_k \quad (\text{if } n = k) \quad \text{or} \quad S_V^k f = \rho_{\alpha_1} \cdots \rho_{\alpha_{n-k}} p_k = q_k \quad (\text{if } n > k). \quad (5.6)$$

For f to be a right common multiple of the minimally possible degree, it is necessary and sufficient that $p_k \equiv 1$, and then f is recovered from (5.6) as in (5.3). The formula (5.4) is justified quite similarly. \square

Corollary 5.3. *Let $g_1, \dots, g_m \in \mathcal{IP}_V$ be such that $\deg(g_1) \geq \deg(g_2) \geq \dots \geq \deg(g_m)$. If g_1, \dots, g_m are pairwise left (right) coprime, then $\text{lrcm}(g_1, g_2, \dots, g_m) = \text{lrcm}(g_1, g_2)$ (respectively, $\text{llcm}(g_1, g_2, \dots, g_m) = \text{llcm}(g_1, g_2)$).*

Proof: Let $\deg(g_j) = d_j$. By Lemma 5.2, $f := \text{lrcm}(g_1, g_2) = \mathcal{X}_V^{d_2} h$ for some $h \in \mathcal{IP}_V$ with $\deg(h) = d_1 - d_2$. Since $g_j g_j^\# = \mathcal{X}_V^{d_j}$, it follows that $\mathcal{X}_V^{d_2}$ is a right common multiple of g_2, \dots, g_m . Thus, f is a right common multiple of g_2, \dots, g_m and the least right common multiple of g_1 and g_2 . Therefore, $f = \text{lrcm}(g_1, g_2, \dots, g_m)$, which proves the first statement. The second statement is verified similarly. \square

By Corollary 4.5, any monic polynomial $f \in \mathcal{P}_V$ can be uniquely represented as the product $f = \mathcal{X}_V^k p$ of a polynomial $p \in \mathcal{IP}_V$ and a power of the characteristic

polynomial \mathcal{X}_V . Formula (5.3) (along with Theorem 5.1) tells us that the $\mathbf{lrcm}(g, h)$ of two relatively prime polynomials $g, h \in \mathcal{IP}_V$ ($\deg(g) \geq \deg(h)$) is of exactly the same form, where $k = \deg(h)$ and p is the (unique) left divisor of g of degree

$$\deg(p) = \deg(g) - \deg(h) = \deg(f) - 2\deg(h).$$

We thus arrive at the following result.

Lemma 5.4. *Any polynomial $f \in \mathcal{P}_V$ with $m_s(V; f) = k \geq 1$ can be represented as*

$$f = \mathbf{lrcm}(g, h), \quad g, h \in \mathcal{IP}_V, \quad \mathcal{Z}_\ell(g) \neq \mathcal{Z}_\ell(h), \quad (5.7)$$

and for any such representation, $\deg(h) = k$ and $\deg(g) = \deg(f) - k$. Moreover,

- (1) If $f = \mathcal{X}_V^k$, then (5.7) holds for any $g, h \in \mathcal{IP}_V$ with $\deg(h) = \deg(g) = k$.
- (2) If $f = \mathcal{X}_V^k p$ with $p = \rho_{\alpha_1} \dots \rho_{\alpha_n} \in \mathcal{IP}_V$, then all pairs (g, h) giving rise to representation (5.7) are characterized by the properties
 - (a) $\deg(h) = k$ and $\mathcal{Z}_\ell(h) \neq \{\alpha_1\}$;
 - (b) $g = pq$ for some $q \in \mathcal{IP}_V$ such that $\deg(q) = k$ and $\mathcal{Z}_\ell(q) \neq \{\bar{\alpha}_n\}$.

Remark 5.5. In Lemma 5.4, one can choose $h = \rho_{\alpha}^k$ and $g = \rho_{\alpha}^k$ for any fixed $\alpha \in V$ in case (1) and $h = \rho_{\alpha_1}^k$ and $q = \rho_{\alpha_n}^k$ in case (2). Observe that these particular choices were used in the proof of Theorem 5.1.

We now formulate a more detailed version of Theorem 1.2. Recall that noncommutative polynomials p_1, \dots, p_n are said to be *left (right) relatively prime* if each polynomial p_k has no common left (right) zeros with the least right (left) common multiple of all other polynomials. We also recall that the spherical divisors D_{ℓ, V_j}^f and $D_{\mathbf{r}, V_j}^f$ of a polynomial f are unique by Theorem 1.1.

Theorem 5.6. *Let $f \in \mathbb{H}[z]$ be a monic polynomial with spherical zeros V_1, \dots, V_m and isolated zeros contained in conjugacy classes V_{m+1}, \dots, V_n . There exist two sets $\Pi = \{p_i\}_{i=1}^{m+n}$ and $\tilde{\Pi} := \{\tilde{p}_i\}_{i=1}^{m+n}$ of relatively prime indecomposable polynomials such that*

$$\langle f \rangle_{\mathbf{r}} = \bigcap_{i=1}^{m+n} \langle p_i \rangle_{\mathbf{r}} \quad \text{and} \quad \langle f \rangle_{\ell} = \bigcap_{i=1}^{m+n} \langle \tilde{p}_i \rangle_{\ell}. \quad (5.8)$$

Representations (5.8) are unique in the following sense:

- (1) For each $j = m+1, \dots, n$, the set Π (resp., $\tilde{\Pi}$) contains exactly one polynomial from \mathcal{IP}_{V_j} (for each $j = m+1, \dots, n$) which is equal to D_{ℓ, V_j}^f (resp., $D_{\mathbf{r}, V_j}^f$).
- (2) For each $j = 1, \dots, m$, the set Π (resp., $\tilde{\Pi}$) contains exactly two polynomials from \mathcal{IP}_{V_j} , and the least right (left) common multiple of these polynomials is equal to D_{ℓ, V_j}^f (resp., $D_{\mathbf{r}, V_j}^f$).

Proof: The existence of representations (5.8) follows from Theorem 1.1 and Lemma 5.4. Let us assume that

$$\langle f \rangle_{\mathbf{r}} = \bigcap_{i=1}^M \langle p_i \rangle_{\mathbf{r}} \quad (5.9)$$

for a relatively prime collection $\Pi = \{p_i : 1 \leq i \leq M\}$ of indecomposable polynomials. In particular, each polynomial $p_i \in \Pi$ is relatively prime with the **lrcm** of all other polynomials in Π . By Corollary 5.3, it follows that Π contains at most two polynomials with zeros in the same conjugacy class. Moreover, if Π contains two polynomials in \mathcal{IP}_{V_j} , then V_j is a spherical zero of f , i.e., $j \in \{1, \dots, m\}$. Otherwise, V_j contains isolated zeros of f , i.e., $j \in \{m+1, \dots, n\}$. Therefore, $M = 2m + (n - m) = m + n$. Let us define $F_j \in \mathcal{P}_{V_j}$ as the **lrcm** of two elements in $\Pi \cap \mathcal{IP}_{V_j}$ for $j = 1, \dots, m$ or as a unique element in $\Pi \cap \mathcal{IP}_{V_j}$ for $j \geq m$. Then we have from (5.9),

$$\langle f \rangle_{\mathbf{r}} = \bigcap_{j=1}^n \langle F_j \rangle_{\mathbf{r}}, \quad F_j \in \mathcal{P}_{V_j}.$$

As we know from Theorem 1.1, the latter representation implies $F_j = D_{\ell, V_j}^f$ for $j = 1, \dots, n$. This completes the proof of the part concerning the first representation in (5.8). The dual part is verified in much the same way. \square

6. LEAST COMMON MULTIPLES

In the two previous sections, we represented a given polynomial $f \in \mathbb{H}[z]$ as the least common multiple of its divisors of certain type. Now we address the converse problem.

Problem 6.1. *Given a finite collection $\Pi = \{g_1, \dots, g_m\} \subset \mathbb{H}[z]$, construct explicitly $\text{lrcm}(g_1, \dots, g_m)$ and $\text{llcm}(g_1, \dots, g_m)$.*

Lemma 5.2 and Corollary 5.3 settled the case where $\Pi \subset \mathcal{IP}_V$ consists of pairwise coprime polynomials. The next result removes the coprimeness assumption. Throughout the section, we will be dealing only with right common multiples and consequently, with the left zero structure. The dual statements are analogous and will be omitted.

Lemma 6.2. *Given $g_1, \dots, g_m \in \mathcal{IP}_V$ such that $\deg(g_1) \geq \deg(g_2) \geq \dots \geq \deg(g_m)$,*

let $g_1 = \prod_{i=1}^{\widehat{n}} \rho_{\alpha_i} := \rho_{\alpha_1} \rho_{\alpha_2} \cdots \rho_{\alpha_n}$ and let

$$g_j = p_j h_j, \quad \text{where } p_j = \mathbf{glcd}(g_j, g_1) \quad \text{for } j = 2, \dots, m. \quad (6.1)$$

Then

$$\text{lrcm}(g_1, g_2, \dots, g_m) = \mathcal{X}_V^k \rho_{\alpha_1} \rho_{\alpha_2} \cdots \rho_{\alpha_{n-k}}, \quad \text{where } k = \max_{2 \leq j \leq m} \deg(h_j). \quad (6.2)$$

Proof: Since g_1 is indecomposable, its left divisor p_j is of the form $p_j = \prod_{i=1}^{\deg(\widehat{p_j})} \rho_{\alpha_i}$, by property (4) in Theorem 5.1. Therefore, for a fixed $j \in \{2, \dots, m\}$, we have

$$\begin{aligned} \text{lrcm}(g_j, g_1) &= p_j \cdot \text{lrcm}\left(h_j, \prod_{i=\deg(p_j)+1}^{\widehat{n}} \rho_{\alpha_i}\right) \\ &= \mathcal{X}_V^{\deg(h_j)} \cdot p_j \cdot \prod_{i=\deg(p_j)+1}^{n-\deg(h_j)} \rho_{\alpha_i} \\ &= \mathcal{X}_V^{\deg(h_j)} \cdot \prod_{i=1}^{n-\deg(h_j)} \rho_{\alpha_i} = \text{lrcm}\left(\rho_{\alpha_1}^{\deg(h_j)}, g_1\right). \end{aligned} \quad (6.3)$$

The first equality in the latter calculation follows from (6.1), the second follows by applying Lemma 5.2 to left coprime polynomials h_j and $\rho_{\alpha_{\deg(p_j)}} \cdots \rho_{\alpha_n}$ and since the polynomial \mathcal{X}_V is real, the third equality follows from factorization of p_j , and the last equality follows from Lemma 5.4 (see also Remark 5.5). We now get (6.2):

$$\begin{aligned} \text{lrcm}(g_1, g_2, \dots, g_m) &= \text{lrcm}(g_1, \text{lrcm}(g_2, g_1), \dots, \text{lrcm}(g_m, g_1)) \\ &= \text{lrcm}\left(g_1, \text{lrcm}\left(\rho_{\alpha_1}^{\deg(h_2)}, g_1\right), \dots, \text{lrcm}\left(\rho_{\alpha_1}^{\deg(h_m)}, g_1\right)\right) \\ &= \text{lrcm}\left(g_1, \rho_{\alpha_1}^{\deg(h_2)}, \dots, \rho_{\alpha_1}^{\deg(h_m)}\right) \\ &= \text{lrcm}\left(g_1, \rho_{\alpha_1}^k\right) = \mathcal{X}_V^k \rho_{\alpha_1} \rho_{\alpha_2} \cdots \rho_{\alpha_{n-k}}, \end{aligned}$$

where the first and the third equalities are self-evident, the second equality holds due to (6.3), the fourth equality holds due to the choice (6.2) of k , and the last equality holds by Lemma 5.2 applied to left coprime polynomials g_1 and $\rho_{\alpha_1}^k$. \square

Observe that if g_2, \dots, g_m are left coprime with g_1 , then $p_j \equiv 1$ for $j = 2, \dots, m$ and Corollary 5.3 follows from Lemma 6.3. We next consider Problem 6.1 for indecomposable polynomials having zeros in distinct conjugacy classes. The case where all polynomials are linear ($\Pi = \{\rho_{\gamma_1}, \dots, \rho_{\gamma_n}\}$), has been known for a while; see e.g., [6]. To handle the general case, we need the following preliminary result.

Lemma 6.3. *Let V be a conjugacy class, let $F = \rho_{\alpha_1} \cdots \rho_{\alpha_k} \in \mathcal{IP}_V$ and let $Q \in \mathbb{H}[z]$ be such that $\mathcal{Z}(Q) \cap V = \emptyset$. Define $\tilde{\alpha}_1, \dots, \tilde{\alpha}_k \in \mathbb{H}$ and $Q_0, Q_1, \dots, Q_k \in \mathbb{H}[z]$ by*

$$Q_0 = Q, \quad \tilde{\alpha}_j = Q_{j-1}^{e_{\ell}}(\alpha_j)^{-1} \alpha_j Q_{j-1}^{e_{\ell}}(\alpha_j), \quad Q_j = L_{\alpha_j}\left(Q_{j-1} \rho_{\tilde{\alpha}_j}\right) \quad (6.4)$$

for $j = 1, \dots, k$. Then $\mathcal{Z}(Q_k) \cap V = \emptyset$ and

$$\text{lrcm}(F, Q) = F Q_k = Q \rho_{\tilde{\alpha}_1} \rho_{\tilde{\alpha}_2} \cdots \rho_{\tilde{\alpha}_k}. \quad (6.5)$$

Proof: Since $\mathcal{Z}(F) \subset V$ and $\mathcal{Z}(Q) \cap V = \emptyset$, it follows (see [15, Proposition 4.2] for the proof) that $\deg(\mathbf{lrcm}(F, Q)) = \deg(F) + \deg(Q) = \deg(F) + k$. Therefore, it suffices to find a right common multiple of polynomials F and Q of degree equal $\deg(F) + k$.

Based on the second statement in Lemma 4.7 and definitions (6.4), an induction argument shows that

$$\rho_{\alpha_j} Q_j = Q_{j-1} \rho_{\tilde{\alpha}_j} \quad \text{and} \quad \mathcal{Z}(Q_j) \cap V = \emptyset \quad \text{for} \quad j = 1, \dots, k.$$

Therefore, we get recursively

$$\begin{aligned} FQ_k &= \rho_{\alpha_1} \rho_{\alpha_2} \cdots \rho_{\alpha_{k-1}} \rho_{\alpha_k} Q_k \\ &= \rho_{\alpha_1} \rho_{\alpha_2} \cdots \rho_{\alpha_{k-1}} Q_{k-1} \rho_{\tilde{\alpha}_k} \\ &= \rho_{\alpha_1} \cdots \rho_{\alpha_{k-2}} Q_{k-2} \rho_{\tilde{\alpha}_{k-1}} \rho_{\tilde{\alpha}_k} = \dots = Q_0 \rho_{\tilde{\alpha}_1} \rho_{\tilde{\alpha}_2} \cdots \rho_{\tilde{\alpha}_k} = Q \rho_{\tilde{\alpha}_1} \rho_{\tilde{\alpha}_2} \cdots \rho_{\tilde{\alpha}_k}, \end{aligned}$$

from which we conclude that the polynomial FQ_k is a right common multiple of F and Q . It is clear that its degree equals $\deg(F) + k$, so that (6.5) follows. \square

The next algorithm produces the \mathbf{lrcm} of n indecomposable polynomials

$$F_i = \rho_{\alpha_{i,1}} \rho_{\alpha_{i,2}} \cdots \rho_{\alpha_{i,k_i}} \in \mathcal{IP}_{V_i} \quad (i = 1, \dots, n) \quad (6.6)$$

with zeros in n distinct conjugacy classes $V_1, \dots, V_n \subset \mathbb{H}$.

Algorithm 6.4. *Given polynomials (6.6),*

- (1) Let $P_1 := F_1 = \rho_{\alpha_{1,1}} \rho_{\alpha_{1,2}} \cdots \rho_{\alpha_{1,k_1}}$ and let $i := 2$.
- (2) Let $Q_0 := P_i$ and perform the recursion

$$\tilde{\alpha}_{i+1,j} = Q_{j-1}^{e_\ell} (\alpha_{i+1,j})^{-1} \alpha_{i+1,j} Q_{j-1}^{e_\ell} (\alpha_{i+1,j}), \quad Q_j = L_{\alpha_{i+1,j}} \left(Q_{j-1} \rho_{\tilde{\alpha}_{i+1,j}} \right) \quad (6.7)$$

for $j = 1, \dots, k_i$.

- (3) Let $P_{i+1} := P_i Q_{k_i}$. If $i < n-1$, then let $i := i+1$ and go to (2). If $i = n-1$, proceed to (4)
- (4) The polynomial P_n is equal to the $\mathbf{lrcm}(F_1, \dots, F_n)$.

To justify the algorithm, let us assume that $P_i = \mathbf{lrcm}(F_1, \dots, F_i)$. Then it follows from Lemma 6.3 that

$$\begin{aligned} P_{i+1} &:= P_i Q_{k_i} = \mathbf{lrcm}(P_i, F_{i+1}) \\ &= \mathbf{lrcm}(\mathbf{lrcm}(F_1, \dots, F_i), F_{i+1}) = \mathbf{lrcm}(F_1, \dots, F_{i+1}). \end{aligned}$$

Since $P_1 = F_1 = \mathbf{lrcm}(F_1)$, we conclude by induction that $P_i = \mathbf{lrcm}(F_1, \dots, F_i)$ for all $i = 1, \dots, n$. \square

Example 6.5. We illustrate Algorithm 6.4 by constructing $\mathbf{lrcm}(\rho_\alpha^2, \rho_\beta^2)$ for two non-real quaternions $\alpha \not\sim \beta$. We let $P_1 = \rho_\alpha^2$, and perform two steps of recursion

(6.7):

$$\begin{aligned}
Q_0 &= P_1 = \rho_\alpha^2, & Q_0^{e\ell}(\beta) &= \beta^2 - 2\beta\alpha + \alpha^2, \\
\beta_1 &= (\beta^2 - 2\beta\alpha + \alpha^2)^{-1}\beta(\beta^2 - 2\beta\alpha + \alpha^2), \\
Q_1 &= L_\beta(\rho_\alpha^2 \rho_{\beta_1}) = z^2 + (\beta - 2\alpha - \beta_1)z + \beta^2 - 2\beta\alpha + 2\alpha\beta_1 - \beta\beta_1 + \alpha^2, \\
Q_1^{e\ell}(\beta) &= 3\beta^2 - 4\beta\alpha + \alpha^2 + 2(\alpha - \beta)\beta_1, \\
\beta_2 &= (3\beta^2 - 4\beta\alpha + \alpha^2 + 2(\alpha - \beta)\beta_1)^{-1}\beta(3\beta^2 - 4\beta\alpha + \alpha^2 + 2(\alpha - \beta)\beta_1).
\end{aligned} \tag{6.8}$$

Now we can write the answer:

$$G(z) := \mathbf{lrcm}(\rho_\alpha^2, \rho_\beta^2) = (z - \alpha)^2(z - \beta_1)(z - \beta_2). \tag{6.9}$$

If we choose in the previous example, $\alpha = \mathbf{i}$ and $\beta = 1 + \mathbf{j}$, then formulas (6.8) give

$$\begin{aligned}
\beta_1 &= (-1 - 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})^{-1}(1 + \mathbf{j})(-1 - 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = 1 - \frac{12\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}}{13}, \\
\beta_2 &= (-21 - 10\mathbf{i} + 50\mathbf{j} + 14\mathbf{k})^{-1}(1 + \mathbf{j})(-21 - 10\mathbf{i} + 50\mathbf{j} + 14\mathbf{k}) \\
&= 1 + \frac{-1588\mathbf{i} + 2645\mathbf{j} + 980\mathbf{k}}{3237},
\end{aligned}$$

and we get the \mathbf{lrcm} of polynomials $(z - \mathbf{i})^2$ and $(z - 1 - \mathbf{j})^2$ using formula (6.9).

We would like to stress that to ensure $m_\ell(\beta; G) = 2$ for G of the form (6.9), we cannot choose $\beta_2 = \beta_1$; actually, it can be shown that $\mathbf{lrcm}(\rho_\alpha^2, \rho_\beta^2) = \rho_\alpha^2 \rho_{\beta_1}^2$ (with β_1 as in (6.8)) if and only if $\alpha\beta = \beta\alpha$ in which case we also have $\beta_1 = \beta$. Thus, finding least common multiples of special irreducible polynomials $\rho_{\alpha_i}^{k_i}$ is the computational problem of about the same complexity as the generic one.

We now present several algorithms based on Algorithm 6.4. The first algorithm constructs a polynomial with prescribed left spherical divisors. The input is the collection of polynomials $F_j \in \mathcal{P}_{V_j}$ ($j = 1, \dots, m$), and the output is $f = \mathbf{lrcm}(F_1, \dots, F_m)$, which, according to Theorem 1.1, is a unique monic polynomial such that

$$\mathcal{Z}(f) \subset \bigcup_{j=1}^m V_j \quad \text{and} \quad D_{\ell, V_j}^f = F_j \quad \text{for} \quad j = 1, \dots, m. \tag{6.10}$$

By Corollary 4.5, each F_j is either of the form $F_j = \rho_{x_j}^{k_j}$ (if $V_j = \{x_j\} \subset \mathbb{R}$) or, otherwise, $F_j = \mathcal{X}_{V_j}^{\kappa_j} p_j$, where either $p_j \equiv 1$ or $p_j \in \mathcal{IP}_{V_j}$.

Algorithm 6.6. Let F_1, \dots, F_m ($F_j \in \mathcal{P}_{V_j}$) be arranged so that

$$F_j = \mathcal{X}_{V_j}^{\kappa_j} p_j \quad (1 \leq j \leq m_1) \quad F_j = \mathcal{X}_{V_j}^{\kappa_j} \quad (m_1 < j \leq m_2); \quad F_j = \rho_{x_j}^{k_j} \quad (m_2 < j \leq m),$$

where $p_j \in \mathcal{IP}_{V_j}$ for $j = 1, \dots, m_1$. Use Algorithm 6.4 to compute $G = \mathbf{lrcm}(p_1, \dots, p_{m_1})$ and write

$$\mathbf{lrcm}(F_1, \dots, F_m) = G \cdot \left(\prod_{j=1}^{m_2} \mathcal{X}_{V_j}^{\kappa_j} \right) \cdot \left(\prod_{j=m_1+1}^{m_2} \rho_{x_j}^{\kappa_j} \right). \tag{6.11}$$

Proof: Algorithm 6.4 applies since p_1, \dots, p_{m_1} are indecomposable polynomials with zeros in m_1 distinct conjugacy classes. The rest is clear since $\rho_{x_j}^{k_j}$ and $\mathcal{X}_{V_j}^{k_j}$ are real polynomials and since $\mathbf{lrcm}(F_1, \dots, F_{m_1}) = \left(\prod_{j=1}^{m_1} \mathcal{X}_{V_j}^{k_j} \right) \cdot \mathbf{lrcm}(p_1, \dots, p_{m_1})$. \square

We next consider the case where all polynomials have zeros in the same (non-real) conjugacy class $V \subset \mathbb{H}$. Without loss of generality we may assume that at most one of given polynomials is real (i.e., of the form \mathcal{X}_V^κ) – if there are several, then removing all of them but the one of highest degree will not affect the \mathbf{lrcm} .

Algorithm 6.7. *Given polynomials $F_1 = \mathcal{X}_V^{\kappa_1}$ and $F_j = \mathcal{X}_V^{\kappa_j} p_j$ ($j = 2, \dots, m$), where $p_j = \rho_{\alpha_{j,1}} \rho_{\alpha_{j,2}} \cdots \rho_{\alpha_{j,k_j}} \in \mathcal{IP}_V$,*

- (1) *Construct the polynomials g_{j1} and g_{j2} as follows:*
 - (a) *Let $g_{11} = \rho_\alpha^{\kappa_1}$ and $g_{12} = \rho_\alpha^{\kappa_1}$ for any fixed $\alpha \in V$.*
 - (b) *For $j = 2, \dots, m$, let $g_{j1} = p_j \rho_{\alpha_{j,k_j}}^{\kappa_j}$ and $g_{j2} = \rho_\alpha^{\kappa_j}$.*
- (2) *Find $f = \mathbf{lrcm}(g_{j1}, g_{j2} : 1 \leq j \leq m)$ as in Lemma 6.2. Then we also have $f = \mathbf{lrcm}(F_1, \dots, F_m)$.*

The statement is obvious since by Remark 5.5, $F_j = \mathbf{lrcm}(g_{j1}, g_{j2})$ for $j = 1, \dots, m$. Note that in Step 2, we may dismiss all polynomials F_j of degree $\deg F_j \leq \kappa_1$ (since F_1 is a multiple of each such polynomial). Also, if the polynomials $p_j \in \mathcal{IP}_V$ are pairwise left coprime, then we may choose α in Step 1(a) so that the polynomials g_{ji} are all pairwise coprime in which case Step 2 simplifies to

2'. *Pick (any) two polynomials in the set $\{g_{j1}, g_{j2} : 1 \leq j \leq m\}$ with highest degrees and find their least right common multiple using formula (5.3).*

We finally present the algorithm that produces the \mathbf{lrcm} of any finite collection of quaternion polynomials. The construction is based on the fact that the left spherical divisor of the $\mathbf{lrcm}(g_1, \dots, g_m)$ associated with a conjugacy class V is equal to the \mathbf{lrcm} of left spherical divisors of g_1, \dots, g_m associated with V .

Algorithm 6.8. *Given polynomials $g_1, \dots, g_m \in \mathbb{H}[z]$,*

- (1) *Find all conjugacy classes V_1, \dots, V_n containing at least one left zero of at least one polynomial from the set.*
- (2) *Use Algorithm 4.3 to find all spherical divisors $D_{\ell, V_i}^{g_j}$ of g_j for $j = 1, \dots, m$. If $\mathcal{Z}(g_j) \cap V_i = \emptyset$, let $D_{\ell, V_i}^{g_j} \equiv 1$.*
- (3) *For each $i = 1, \dots, n$, use Algorithm 6.7 to construct $F_i = \mathbf{lrcm}(D_{\ell, V_i}^{g_1}, \dots, D_{\ell, V_i}^{g_m})$.*
- (4) *Use Algorithm 6.6 to construct $f = \mathbf{lrcm}(F_1, \dots, F_n)$.*

Then we also have $f = \mathbf{lrcm}(g_1, \dots, g_m)$.

7. FORMAL POWER SERIES OVER QUATERNIONS

As in the commutative case, certain results concerning quaternion polynomials can be extended to formal power series over \mathbb{H} (see e.g., [7], [10], [1]). We are particularly interested in power series for which left and right evaluation functionals

make sense. We denote by $\mathbb{B} = \{\alpha \in \mathbb{H} : |\alpha| < 1\}$ the open unit ball in \mathbb{H} , and introduce the ring

$$\mathcal{H} = \left\{ f(z) = \sum_{j=0}^{\infty} f_j z^j : \limsup \sqrt[k]{|f_k|} \leq 1 \right\}.$$

Observe that for any $f(z) = \sum f_k z^k$ in \mathcal{H}_R and any $\alpha \in \mathbb{B}$, the quaternion series $\sum_{k=0}^{\infty} \alpha^k f_k$ and $\sum_{k=0}^{\infty} f_k \alpha^k$ converge absolutely, so the evaluation formulas (2.2) (with $m = \infty$) and therefore, the notions of left and right zeros (within \mathbb{B}) make sense. Furthermore, the power series $L_\alpha f$ and $R_\alpha f$ (defined as in (2.3) but with $m = \infty$) are also in \mathcal{H} . Equalities (2.1) and therefore, equivalences (1.5) hold true in \mathcal{H} as well as evaluation formulas (2.11). The conjugate power series f^\sharp is defined as in (2.6) (with $m = \infty$) and relations (2.7) hold for $f, g \in \mathcal{H}$. Moreover, the real power series $ff^\sharp \in \mathbb{R}[[z]]$ belongs to \mathcal{H} and (by the complex uniqueness theorem) has countably many spherical zeroes in \mathbb{B} of finite multiplicity each. Theorem 2.2 extends to \mathcal{H} as follows: *if $f \in \mathcal{H}$, then each conjugacy class $V \subset \mathcal{Z}(ff^\sharp)$ either contains exactly one left and one right zero of f or $V \subset \mathcal{Z}_\ell(f) \cap \mathcal{Z}_r(f)$.*

Remark 7.1. *Theorem 3.1 holds for any $f \in \mathcal{H}$.*

Indeed, formulas (3.1) rely on representation formulas (3.2) and (3.3), which hold true for monomials $f_j z^j$ and therefore, for elements in \mathcal{H} , by linearity. Furthermore, since $m_s([\alpha], ff^\sharp)$ is finite for every $\alpha \in \mathbb{B}$, Algorithms 4.1 and 4.3 apply to $f \in \mathcal{H}$; the only modification is that Q_j and \tilde{Q}_j obtained via recursions (4.2)-(4.5) are power series from \mathcal{H} rather than polynomials. Algorithm 4.1 recovers in a constructive way the following result from [7].

Proposition 7.2. *Given $f \in \mathcal{H}$, let V be a conjugacy class such that $m_s(V, ff^\sharp) = k$. There exist (unique) integer $\kappa \geq 0$, spherical chains $\alpha = (\alpha_1, \dots, \alpha_{k-2\kappa}) \subset V$ and $\tilde{\alpha} = (\tilde{\alpha}_1, \dots, \tilde{\alpha}_{k-2\kappa}) \subset V$, and power series $P, \tilde{P} \in \mathcal{H}$ having no zeros in V such that*

$$f = \mathcal{X}_V^\kappa \rho_{\alpha_1} \rho_{\alpha_2} \cdots \rho_{\alpha_{k-2\kappa}} P = \tilde{P} \rho_{\tilde{\alpha}_{k-2\kappa}} \cdots \rho_{\tilde{\alpha}_2} \rho_{\tilde{\alpha}_1} \mathcal{X}_V^\kappa.$$

Although Algorithms 4.8 and 4.10 relating left and right zero structures apply to formal power series, they might be efficient only if evaluations f^{e_ℓ} and f^{e_r} admit closed formulas.

Remark 7.3. *Proposition 7.2 allows us to introduce spherical divisors of a given $f \in \mathcal{H}$ (which are polynomials uniquely determined from f). Constructing an $f \in \mathcal{H}$ subject to conditions (6.10) (i.e., with finitely many prescribed spherical divisors D_{ℓ, V_i}^f) reduces to Algorithm 6.6 since any such f is of the form $f = GH$ where $G = \text{lrcm}(D_{\ell, V_1}^f, \dots, D_{\ell, V_m}^f)$ and $H \in \mathcal{H}$ has no zeros in \mathbb{B} . The latter construction does not seem particularly interesting unless we desire to construct an f with some extra properties; then the relevant question is how to choose H to achieve this. In the concluding section we consider a problem of this sort.*

7.1 Finite Blaschke products with prescribed zero structure. We denote by

$$\mathbb{H}^2 = \left\{ h(z) = \sum_{j=0}^{\infty} z^j h_j : \|h\|_{\mathbb{H}^2}^2 := \sum_{j=0}^{\infty} |h_j|^2 < \infty \right\}$$

the space of elements in $\mathbb{H}[[z]]$ with square summable coefficients and observe that $\mathbb{H}^2 \subset \mathcal{H}$. An element $f \in \mathbb{H}^2$ is called *bi-inner* if the equalities

$$\|f \cdot h\|_{\mathbb{H}^2} = \|h\|_{\mathbb{H}^2} = \|h \cdot f\|_{\mathbb{H}^2} \quad \text{for all } h \in \mathbb{H}^2 \quad (7.1)$$

hold. If f satisfies only the first (only the second) equality in (7.1), it is called *left-inner* (*right-inner*). It is easily seen that f is left-inner if and only if $f^\#$ is right-inner.

For each fixed $\alpha \in \mathbb{B}$, the power series

$$\mathbf{k}_\alpha(z) = \sum_{k=0}^{\infty} \alpha^k z^k \quad \text{belongs to } \mathbb{H}^2 \text{ and } \|\mathbf{k}_\alpha\|_{\mathbb{H}^2}^2 = \frac{1}{1-|\alpha|^2}. \quad (7.2)$$

Letting $\Upsilon_{\alpha,\gamma} := 1 - (\alpha + \bar{\alpha})\gamma + |\alpha|^2\gamma^2$ and observing equalities

$$\Upsilon_{\alpha,\gamma} \cdot \left(\sum_{k=0}^{\infty} \gamma^k \alpha^k \right) = 1 - \gamma \bar{\alpha} \quad \text{and} \quad \left(\sum_{k=0}^{\infty} \alpha^k \gamma^k \right) \cdot \Upsilon_{\alpha,\gamma} = 1 - \bar{\alpha} \gamma \quad (|\gamma| < 1),$$

we come up with evaluation formulas

$$\mathbf{k}_\alpha^{e\ell}(\gamma) = \Upsilon_{\alpha,\gamma}^{-1} \cdot (1 - \gamma \bar{\alpha}), \quad \mathbf{k}_\alpha^{er}(\gamma) = (1 - \bar{\alpha} \gamma) \cdot \Upsilon_{\alpha,\gamma}^{-1} \quad \text{for all } \gamma \in \mathbb{B}, \quad (7.3)$$

which imply in particular, that \mathbf{k}_α has no zeros in \mathbb{B} . Since $\mathbf{k}_\alpha(z) \cdot (1 - z\alpha) \equiv 1$, the power series \mathbf{k}_α is the formal inverse of the polynomial $1 - z\alpha$. Hence, the power series

$$\mathbf{b}_\alpha(z) := \boldsymbol{\rho}_\alpha(z) \cdot \mathbf{k}_{\bar{\alpha}}(z) = (z - \alpha) \cdot \sum_{k=0}^{\infty} \bar{\alpha}^k z^k = -\alpha + (1 - |\alpha|^2)z \cdot \mathbf{k}_{\bar{\alpha}}(z) \quad (7.4)$$

can be viewed as the quaternionic analog of the Blaschke factor $\frac{z-\alpha}{1-\bar{\alpha}z}$. It is seen from the second representation in (7.4) that $\mathcal{Z}_\ell(\mathbf{b}_\alpha) = \mathcal{Z}_r(\mathbf{b}_\alpha) = \{\alpha\}$. In analogy to the complex case, \mathbf{b}_α gives rise to two automorphisms $\gamma \mapsto \mathbf{b}_\alpha^{e\ell}(\gamma)$ and $\gamma \mapsto \mathbf{b}_\alpha^{er}(\gamma)$ of the closed unit ball $\bar{\mathbb{B}}$ (see [12], [3]). The latter can be derived from the evaluation formulas

$$\mathbf{b}_\alpha^{e\ell}(\gamma) = \Upsilon_{\alpha,\gamma}^{-1} \cdot (\gamma(\alpha^2 + 1) - (\gamma^2 + 1)\alpha), \quad \mathbf{b}_\alpha^{er}(\gamma) = ((\alpha^2 + 1)\gamma - \alpha(\gamma^2 + 1)) \cdot \Upsilon_{\alpha,\gamma}^{-1}$$

which, in turn, follow from (7.3) (with $\bar{\alpha}$ instead of α) and (7.1).

Proposition 7.4. *For each $\alpha \in \mathbb{B}$, the power series \mathbf{b}_α is bi-inner.*

Proof: The left-inner property \mathbf{b}_α has been observed in [2]. Our justification is similar to that in [2] but does not rely on the Hilbert space structure of \mathbb{H}^2 . We first

verify that $\|\mathbf{b}_\alpha \cdot h\|_{\mathbb{H}^2} = \|h\|_{\mathbb{H}^2}$ for every h of the form $h(z) = d + cz^k$. Since

$$\begin{aligned} \mathbf{b}_\alpha(z) \cdot h(z) &= -\alpha d + (1 - |\alpha|^2) \cdot \sum_{j=1}^{k-1} \bar{\alpha}^{j-1} dz^j + ((1 - |\alpha|^2)\bar{\alpha}^{k-1}d - \alpha c)z^k \\ &\quad + (1 - |\alpha|^2) \cdot \sum_{j=0}^{\infty} \bar{\alpha}^j (\bar{\alpha}^k d + c)z^{j+k+1}, \end{aligned}$$

we have, by the definition of the \mathbb{H}^2 -norm,

$$\begin{aligned} \|\mathbf{b}_\alpha \cdot h\|_{\mathbb{H}^2}^2 &= |\alpha|^2 |d|^2 + (1 - |\alpha|^2) \cdot (1 - |\alpha|^{2k-2}) |d|^2 + |(1 - |\alpha|^2)\bar{\alpha}^{k-1}d - \alpha c|^2 \\ &\quad + (1 - |\alpha|^2) \cdot |\bar{\alpha}^k d + c|^2 \\ &= |d|^2 + |c|^2 + (1 - |\alpha|^2) \cdot \left(2\operatorname{Re}(\bar{\alpha}^k d \bar{c}) - 2\operatorname{Re}(\bar{\alpha}^{k-1} d \bar{c} \bar{\alpha}) \right). \end{aligned} \quad (7.5)$$

Applying the well-known quaternionic equality $\operatorname{Re}(ab) = \operatorname{Re}(ba)$ to $a = \bar{\alpha}$ and $b = \bar{\alpha}^{k-1} d \bar{c}$, we conclude from (7.5) that $\|\mathbf{b}_\alpha \cdot h\|_{\mathbb{H}^2}^2 = |c|^2 + |d|^2 = \|h\|_{\mathbb{H}^2}^2$ for $h(z) = d + cz^k$. Since multiplication by z preserves the \mathbb{H}^2 -norm, it follows that $\|\mathbf{b}_\alpha \cdot h\|_{\mathbb{H}^2} = \|h\|_{\mathbb{H}^2}$ holds for all h of the form $h(z) = dz^\ell + cz^k$ and therefore, for all $h \in \mathbb{H}[z]$. By the limit argument, the equality holds for every $h \in \mathbb{H}^2$ and therefore \mathbf{b}_α is left-inner. Then $\mathbf{b}_{\bar{\alpha}} = \mathbf{b}_\alpha^\#$ is also left-inner, so that \mathbf{b}_α is right-inner and therefore, bi-inner. \square

It is clear that the power series (following [2], we will call it *a finite Blaschke product*)

$$B(z) = \mathbf{b}_{\alpha_1}(z) \mathbf{b}_{\alpha_2}(z) \cdots \mathbf{b}_{\alpha_m}(z) \phi, \quad (\alpha_1, \dots, \alpha_m \in \mathbb{B}, |\phi| = 1) \quad (7.6)$$

is bi-inner. Its zeros are contained in the union of conjugacy classes $[\alpha_1], \dots, [\alpha_m]$ and we can use Algorithm 4.1 to construct the spherical divisors of B corresponding to each class. Now we will consider the question related to Algorithm 6.6: *to construct a bi-inner power series with the prescribed (say, left) zero structure*. By Remark 7.3, in case only finitely many conjugacy classes are involved, the latter question reduces to the following: *given a polynomial*

$$G = \rho_{\alpha_1} \rho_{\alpha_2} \cdots \rho_{\alpha_m}, \quad \alpha_1, \dots, \alpha_m \in \mathbb{B}, \quad (7.7)$$

find a power series $H \in \mathbb{H}^2$ with no zeros in \mathbb{B} so that $f = G \cdot H$ be inner.

In case $\alpha_i \alpha_j = \alpha_j \alpha_i$ for $i, j = 1, \dots, m$ (i.e., $\alpha_1, \dots, \alpha_m$ belong to the same two-dimensional subspace of \mathbb{H}), we can take $H := \mathbf{k}_{\bar{\alpha}_1} \mathbf{k}_{\bar{\alpha}_2} \cdots \mathbf{k}_{\bar{\alpha}_m}$. Some reduction is possible if G has a real zero x (i.e., $G = \rho_x \cdot \tilde{G}$) or a spherical zero $[\alpha]$ ($G = \mathcal{X}_{[\alpha]} \cdot \tilde{G}$). In both cases, it suffices to find \tilde{H} so that $\tilde{G} \cdot \tilde{H}$ is inner and then let $H = \mathbf{k}_x \cdot \tilde{H}$ (in the first case) or $H = \mathbf{k}_\alpha \cdot \mathbf{k}_{\bar{\alpha}} \cdot \tilde{H}$ (in the second case). It thus turns out that the general problem reduces to the one where G does not have real or spherical zeros. The case where $\alpha_1, \dots, \alpha_m$ are pairwise non-equivalent (i.e., has m simple roots) has been handled in [2]. The next theorem settles the general case.

Theorem 7.5. *Given G as in (7.7), there exist β_1, \dots, β_m ($\beta_i \in [\alpha_i]$) so that the power series $G \cdot \mathbf{k}_{\beta_1} \cdot \mathbf{k}_{\beta_2} \cdots \mathbf{k}_{\beta_m}$ is a finite Blaschke product.*

Proof: We will assume (without loss of generality) that $\alpha_1, \dots, \alpha_m$ are all non-real and prove the statement by induction. For $m = 1$, we let $\beta_1 = \overline{\alpha_1}$ and the statement follows by Proposition 7.4. Assume that we have found $\gamma_i, \delta_i \in [\alpha_i]$ ($i = 1, \dots, m-1$) so that

$$\rho_{\alpha_1} \cdot \rho_{\alpha_2} \cdots \rho_{\alpha_{m-1}} \cdot \mathbf{k}_{\delta_1} \cdot \mathbf{k}_{\delta_2} \cdots \mathbf{k}_{\delta_{m-1}} = \mathbf{b}_{\gamma_1} \cdot \mathbf{b}_{\gamma_2} \cdots \mathbf{b}_{\gamma_{m-1}} \varphi. \quad (7.8)$$

Since evaluation functional at real points is multiplicative, evaluating both sides in (7.8) at zero gives

$$\varphi = (\gamma_1 \gamma_2 \cdots \gamma_{m-1})^{-1} (\alpha_1 \alpha_2 \cdots \alpha_{m-1}),$$

and since $\gamma_i \in [\alpha_i]$, it follows that $|\varphi| = 1$. Given equality (7.8) and given α_m , we want to find $\beta_i \in [\alpha_i]$ ($i = 1, \dots, m$) and $\gamma_m \in [\alpha_m]$ such that

$$\rho_{\alpha_1} \cdot \rho_{\alpha_2} \cdots \rho_{\alpha_{m-1}} \cdot \rho_{\alpha_m} \cdot \mathbf{k}_{\beta_1} \cdot \mathbf{k}_{\beta_2} \cdots \mathbf{k}_{\beta_m} = \mathbf{b}_{\gamma_1} \cdot \mathbf{b}_{\gamma_2} \cdots \mathbf{b}_{\gamma_{m-1}} \cdot \mathbf{b}_{\gamma_m} \psi, \quad (7.9)$$

where the unimodular factor ψ is given by $\psi = \gamma_m^{-1} \varphi \alpha_m$. Multiplying both parts in (7.8) by $\varphi^{-1} \mathbf{b}_{\gamma_m} \psi$ on the right and comparing the resulting equality with (7.9), we see that (7.9) is equivalent to

$$\rho_{\alpha_1} \cdots \rho_{\alpha_{m-1}} \cdot \mathbf{k}_{\delta_1} \cdots \mathbf{k}_{\delta_{m-1}} \varphi^{-1} \mathbf{b}_{\gamma_m} \psi = \rho_{\alpha_1} \cdots \rho_{\alpha_{m-1}} \cdot \rho_{\alpha_m} \cdot \mathbf{k}_{\beta_1} \cdots \mathbf{k}_{\beta_m},$$

which in turn, is equivalent to

$$\mathbf{k}_{\delta_1} \cdots \mathbf{k}_{\delta_{m-1}} \varphi^{-1} \mathbf{b}_{\gamma_m} \psi = \rho_{\alpha_m} \cdot \mathbf{k}_{\beta_1} \cdots \mathbf{k}_{\beta_m}. \quad (7.10)$$

The construction of β_1, \dots, β_m , γ_m and ψ subject to identity (7.10) is as follows: letting $\alpha'_1 = \alpha_m$ and $\phi_1 = 1$, we compute α'_k and ϕ_k by the recursive formulas

$$\alpha'_{k+1} = (1 - \overline{\alpha'_k} \phi_k^{-1} \delta_k \phi_k) \cdot \alpha'_k \cdot (1 - \overline{\alpha'_k} \phi_k^{-1} \delta_k \phi_k)^{-1}, \quad (7.11)$$

$$\phi_{k+1} = \phi_k \cdot (1 - \phi_k^{-1} \delta_k \phi_k \overline{\alpha'_k}) (1 - \overline{\alpha'_k} \phi_k^{-1} \delta_k \phi_k)^{-1}, \quad (7.12)$$

for $k = 2, \dots, m-1$. Then we let

$$\beta_k = (1 - \overline{\alpha'_k} \phi_k^{-1} \delta_k \phi_k) \cdot \phi_k^{-1} \delta_k \phi_k \cdot (1 - \overline{\alpha'_k} \phi_k^{-1} \delta_k \phi_k)^{-1} \quad (k = 1, \dots, m-1), \quad (7.13)$$

$$\beta_m = \overline{\alpha'_m}, \quad \psi = \varphi \phi_m, \quad \gamma_m = \psi \alpha'_m \psi^{-1}. \quad (7.14)$$

It follows from (7.11) that $\alpha'_{k+1} \sim \alpha'_k$ for all $k = 1, \dots, m-1$ and in particular, $\alpha'_m \sim \alpha'_1 = \alpha_m$. By (7.14), we also have $\beta_m, \gamma_m \in [\alpha_m]$. From (7.13) and the induction hypothesis we conclude that $\beta_k \sim \delta_k \in [\alpha_k]$ for $k = 1, \dots, m-1$. By (7.12), $|\phi_{k+1}| = |\phi_k|$ for $k = 1, \dots, m-1$ and therefore, $|\phi_m| = 1$. We also conclude from (7.13) that

$$\phi_m \rho_{\alpha'_m} \cdot \mathbf{k}_{\beta_m} = \phi_m \mathbf{b}_{\alpha'_m} = \varphi^{-1} \psi \mathbf{b}_{\alpha'_m} = \varphi^{-1} \mathbf{b}_{\gamma_m} \psi. \quad (7.15)$$

To show that (7.10) holds we first verify equalities

$$\phi_k \alpha'_k = \phi_{k+1} \alpha'_{k+1}, \quad \delta_k \phi_k = \phi_{k+1} \beta_k, \quad \phi_k + \delta_k \phi_k \alpha'_k = \phi_{k+1} (1 + \alpha'_{k+1} \beta_k) \quad (7.16)$$

for $k = 1, \dots, m-1$. Indeed, letting for short $\Delta_k = 1 - \overline{\alpha'_k} \phi_k^{-1} \delta_k \phi_k$, we have from (7.11)–(7.13),

$$\begin{aligned}
\phi_{k+1} \alpha'_{k+1} &= \phi_k \cdot (1 - \phi_k^{-1} \delta_k \phi_k \overline{\alpha'_k}) \alpha'_k \Delta_k^{-1} = \phi_k \alpha'_k \Delta_k \Delta_k^{-1} = \phi_k \alpha'_k, \\
\phi_{k+1} \beta_k &= \phi_k \cdot (1 - \phi_k^{-1} \delta_k \phi_k \overline{\alpha'_k}) \phi_k^{-1} \delta_k \phi_k \Delta_k^{-1} = \phi_k \phi_k^{-1} \delta_k \phi_k \Delta_k \Delta_k^{-1} = \delta_k \phi_k, \\
\phi_{k+1} (1 + \alpha'_{k+1} \beta_k) &= \phi_k \cdot (1 - \phi_k^{-1} \delta_k \phi_k \overline{\alpha'_k}) (\Delta_k^{-1} (1 + \Delta_k \alpha'_k \phi_k^{-1} \delta_k \phi_k \Delta_k^{-1})) \\
&= (\phi_k - \delta_k \phi_k \overline{\alpha'_k}) (1 + \alpha'_k \phi_k^{-1} \delta_k \phi_k) \Delta_k^{-1} \\
&= \phi_k - \delta_k \phi_k \overline{\alpha'_k} + (\phi_k - \delta_k \phi_k \overline{\alpha'_k}) (\alpha'_k + \overline{\alpha'_k}) \phi_k^{-1} \delta_k \phi_k \Delta_k^{-1} \\
&= \phi_k - \delta_k \phi_k \overline{\alpha'_k} + (\alpha'_k + \overline{\alpha'_k}) \delta_k \phi_k \Delta_k \Delta_k^{-1} \\
&= \phi_k - \delta_k \phi_k \overline{\alpha'_k} + \delta_k \phi_k (\alpha'_k + \overline{\alpha'_k}) = \phi_k + \delta_k \phi_k \alpha'_k.
\end{aligned}$$

The consequence of relations (7.16) is the polynomial identity

$$(1 - z \delta_k) \phi_k (z - \alpha'_k) = \phi_{k+1} (z - \alpha'_{k+1}) (1 - z \beta_k),$$

which being multiplied by $\mathbf{k}_{\delta_k}(z)$ on the left and by $\mathbf{k}_{\beta_k}(z)$ on the right, implied

$$\phi_k \rho_{\alpha'_k} \mathbf{k}_{\beta_k} = \mathbf{k}_{\delta_k} \phi_{k+1} \quad \text{for } k = 1, \dots, m-1.$$

Combining the latter equalities with (7.15) gives

$$\begin{aligned}
\rho_{\alpha_m} \cdot \mathbf{k}_{\beta_1} \cdot \mathbf{k}_{\beta_2} \cdot \mathbf{k}_{\beta_3} \cdots \mathbf{k}_{\beta_m} &= \phi_1 \rho_{\alpha'_1} \cdot \mathbf{k}_{\beta_1} \cdot \mathbf{k}_{\beta_2} \cdot \mathbf{k}_{\beta_3} \cdots \mathbf{k}_{\beta_m} \\
&= \mathbf{k}_{\delta_1} \phi_2 \rho_{\alpha'_2} \cdot \mathbf{k}_{\beta_2} \cdot \mathbf{k}_{\beta_3} \cdots \mathbf{k}_{\beta_m} \\
&= \mathbf{k}_{\delta_1} \cdot \mathbf{k}_{\delta_2} \phi_3 \rho_{\alpha'_3} \cdot \mathbf{k}_{\beta_3} \cdots \mathbf{k}_{\beta_m} = \dots \\
&= \mathbf{k}_{\delta_1} \cdot \mathbf{k}_{\delta_2} \cdots \mathbf{k}_{\delta_{m-1}} \phi_m \rho_{\alpha'_m} \cdot \mathbf{k}_{\beta_m} \\
&= \mathbf{k}_{\delta_1} \cdots \mathbf{k}_{\delta_{m-1}} \varphi^{-1} \mathbf{b}_{\gamma_m} \psi
\end{aligned}$$

which confirms (7.10) and completes the proof of the theorem. \square

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DEPARTMENT OF MATHEMATICS, COLLEGE OF WILLIAM AND MARY, WILLIAMSBURG, VA 23187-8795, USA

E-mail address: vladi@math.wm.edu